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ON THE ALGEBRAIC STRUCTURE OF TWISTOR SPACES

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Introduction

The twistor space associated to a compact self-dual 4-manifold is a compact complex 3-fold whose complex structure is determined by the self-dual conformal structure of the 4-manifold. The most characteristic property of a twistor space is that it is foliated by a four-real-parameter family of rational curves with normal bundle isomorphic to that of a line in the complex projective 3-space; indeed, the leaf-space of this foliation is precisely the associated self-dual 4-manifold [2]. The simplest example of a compact nonflat self-dual 4-manifold is the Euclidean 4-sphere; the corresponding twistor space is the complex projective 3-space. A second well-known example is the full-flag space of $\mathbf{C}^3$ as the twistor space associated to the complex projective plane $\mathbf{P}^2$ equipped with the Fubini-Study metric. As shown by Hitchin [10], the preceding two twistor spaces are the only Kählerian twistor spaces, and one might be tempted to believe that methods of algebraic geometry would therefore be of no avail in the study of self-dual manifolds. However, there exist other twistor spaces that are bimeromorphic to algebraic varieties, i.e., Moishezon spaces. The first such examples of this type were described in [18], and correspond to self-dual metrics on the connected-sum of two complex projective planes $\mathbf{P}^2 \# \mathbf{P}^2$. There is in fact a 1-parameter moduli space of such metrics, and each of the corresponding twistor spaces is a small resolution of the intersection of two quadrics in $\mathbf{P}^5$ with four ordinary double points.

At this point, one might ask whether one can find *other* Moishezon twistor spaces. It turns out ([19], [4]) that the 4-manifold associated with such a twistor space must be homeomorphic to an iterated connected-sum $\tau\mathbf{P}^2 := \mathbf{P}^2 \# \cdots \# \mathbf{P}^2$ of τ copies of the complex projective plane and the self-dual conformal class contains a metric of positive scalar curvature. A most encouraging sign was therefore given by the result of Donald-

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son and Friedman [7], proving the existence of self-dual metrics on $\tau\mathbf{P}^2$. LeBrun [13] then gave an explicit construction of *some* self-dual metrics on $\tau\mathbf{P}^2$, showing at the same time by explicit construction that the associated twistor spaces are Moishezon. In fact, they are bimeromorphic to a fiber bundle on a quadric surface.

In LeBrun's construction, he explicitly produces asymptotically Euclidean scalar-flat Kähler metrics on the blow-up of $\mathbf{C}^2$ at collinear points. An orientation reversing one-point compactification of the blow-up of $\mathbf{C}^2$ yields a self-dual metric of positive scalar curvature on a connected-sum $\tau\mathbf{P}^2$, where τ is the number of blowing-up, while the holomorphic compactification of the blow-up of $\mathbf{C}^2$ by adding a copy of $\mathbf{P}^1$ at infinity defines an effective divisor D in the compact twistor space Z associated to $\tau\mathbf{P}^2$. Such a divisor is an example of what we shall call an elementary divisor [15, Definition 2]. We shall herein study the following natural question: how typical or how special are these "*LeBrun twistor spaces*"? In particular, what are the possible values of the algebraic dimension for a twistor space of $\tau\mathbf{P}^2$ equipped with a self-dual metric of positive scalar curvature? For simplicity, we shall restrict our attention to the case when the twistor space contains effective elementary divisor. In a second article [14], LeBrun proved by deformation theory that such twistor spaces exist in abundance, and give rise to asymptotically Euclidean scalar-flat Kähler metrics on τ-fold blow-ups of $\mathbf{C}^2$. The main thrust of our results will be that algebraic dimension depends on whether the blown-up points in question are in relatively special or in relatively general position. (For previous results concerning the algebraic dimension of twistor spaces, see [17], [21], and [22].)

An important feature of an effective elementary divisor is that, when $\overline{D}$ is the conjugate divisor with respect to the real structure on the twistor spaces [2],

$$D + \overline{D} = -\tfrac{1}{2}K,$$

where K is the canonical class on the twistor spaces. This will prove to be of crucial importance for us, because the meromorphic function field is the field of fractions that are homogeneous of degree zero in the graded ring $\bigoplus^n H^0(Z, \mathscr{K}^{-n/2})$ ([15, Proposition 2.3], [19]).

Given this observation and the fact that the Chern number $c_1^3(Z)$ of the twistor space is equal to $16(4-\tau)$, one should not be surprised that we have to study the algebraic structure of the twistor spaces according to the cases when $\tau \leq 3$, $\tau = 4$ and $\tau \geq 5$. As previously noted, both the cases of $\tau = 0$ and $\tau = 1$ are very well known as the first case corresponds to the conformal geometry of a Euclidean 4-sphere and the second case corresponds to the conformal geometry of the Fubini-Study metric on the complex projective plane. The case of $\tau = 2$ is also completely understood [18]. In this case, the twistor space is a small resolution of the intersection of quadrics in $\mathbf{P}^5$. As we shall see in §2, the generic twistor space can also be completely understood when $\tau = 3$. Material in this section is mainly an improvement of §3 of the author's unpublished early work [20]. §3 then gives a characterization of LeBrun's twistor spaces. The results in §§2 and 3 can be combined to give an algebraic description of any twistor space of $3\mathbf{P}^2$ equipped with a self-dual metric of positive scalar curvature. In §§4, 5 and 6, we develop a method to calculate the algebraic dimension of any twistor space of $\tau\mathbf{P}^2$, $\tau \geq 5$, and admitting an effective elementary divisor. §7 is devoted to study the twistor spaces of $4\mathbf{P}^2$. §1 is a preliminary investigation on the structures of the fundamental divisors and elementary divisors. The key technical tricks in this paper are contained in Lemmata (1.9) and (1.10). One of the most important observations is that any effective elementary divisor in the twistor space of $\tau\mathbf{P}^2$ contains a real twistor line L such that the associated map of the complete linear system of L, as a divisor on the surface D, is a blowing-down map from D onto a complex projective plane. In the following summary of results, we shall present the relation between the configuration of this blow-up of $\mathbf{P}^2$ and the algebraic dimension, $a(Z)$, of the twistor space Z.

The main results of this paper are summarized as follows:

Theorem 2.1. *The twistor space associated to a generic self-dual conformal class containing metric of positive scalar curvature on* $3\mathbf{P}^2$ *is a small resolution of the double covering of* $\mathbf{P}^3$ *branched over a quartic with thirteen ordinary double points. Exactly one of these points is real. In homogeneous coordinates* $\{z_0, z_1, z_2, z_3\}$ *on* $\mathbf{P}^2$, *the equation of the quartic is*

$$B(z) = z_0 z_1 z_2 z_3 - Q^2(z),$$

where Q *is a real positive definite quadric.*

Theorem 3.1. *Suppose that the twistor space of a self-dual manifold of positive scalar curvature contains an effective elementary divisor* D. *If the dimension of the complete linear system of the elementary divisor* D *is at least one, then the metric is contained in LeBrun's self-dual conformal class. Equivalently, the twistor space is a LeBrun twistor space. In particular, the twistor space is Moishezon.*

Theorem 7.11. *When the elementary divisor* D *is* $\mathbf{P}^2$ *blown-up at four points so that three of them are collinear, then* $a(Z) = 3$.

Theorem 7.2. *When the elementary divisor D is $\mathbf{P}^2$ blown-up at four points in general position,* $a(Z) \leq 2$.

Theorem 4.2. *If the elementary divisor D is $\mathbf{P}^2$ blown-up at τ points such that all but the last point is collinear, when $\tau \geq 5$,* $a(Z) = 1$.

Theorem 5.6. *If the elementary divisor D is $\mathbf{P}^2$ blown-up at τ points, $\tau \geq 5$, such that all points are co-conic noncollinear, then* $a(Z) \leq 1$.

Theorem 6.3. *If the elementary divisor D is a blow-up of $\mathbf{P}^2$ at τ points, $\tau \geq 6$, such that no three of them are collinear and no six of them are co-conic, then* $a(Z) = 0$.

With the calculations herein, straightforward applications of known results ([4], [14]) immediately lead to the following conclusion: *Fujiki's class $\mathscr{C}$ of complex manifolds, i.e., manifolds bimeromorphic to compact Kähler manifolds, is unstable under small deformations of complex structure.* We defer more detailed discussion of the argument to a forthcoming article [15], as the main results of the present article may be considered to be of independent interest. Interested readers can also find other applications of the results presented here in [16].

Throughout this paper, we shall use a capital letter to denote a divisor, the corresponding boldface letter to denote the associated line bundle and the script letter to denote the sheaf of germs of sections of the associated line bundle. For example, when K is the canonical class of the space Z, $\mathbf{K}$ is the canonical bundle and both $\mathscr{O}(\mathbf{K})$ and $\mathscr{K}$ denote the sheaf of germs of sections of the canonical bundle. When S is a subvariety of the space Z, the notation $\mathbf{K}_{|S}$ denotes the restriction of the canonical bundle of the space Z onto S. It should not be confused with the canonical bundle of the subvariety, which is denoted by $\mathbf{K}_S$.

As usual, the kth cohomology on Z with coefficients in a sheaf $\mathscr{L}$ is denoted by $H^k(Z, \mathscr{L})$. The complex dimension of this vector space is denoted by $h^k(Z, \mathscr{L})$.

1. Preliminaries

If X is a simply connected compact self-dual manifold with positive scalar curvature, a Bochner-type argument shows that the intersection form is positive definite. After Donaldson and Friedman, one can see that the space X is homeomorphic to either a connected-sum of complex projective planes or a 4-sphere. In particular, the topological type of X is completely determined by its signature τ. Associated to the self-dual conformal class on X is the twistor space Z. It is a compact complex manifold of complex dimension 3. As a smooth manifold, Z is the sphere bundle of the anti-self-dual 2-forms on the manifold X. The complex structure on the twistor space is determined by the conformal geometry on X. The projection from Z onto X is called the twistor fibration. The fibers of the twistor fibration are Riemann spheres holomorphically embedded in the twistor space. These holomorphic curves in the twistor space are the *twistor lines.* The Chern numbers of the twistor space are given in terms of the signature of X, τ, as follows [10]:

$$c_1^3 = 16(4-\tau), \quad c_1c_2 = 24, \quad c_3 = 2(2+\tau).$$

The anticanonical bundle $\mathbf{K}^{-1}$ on the twistor space has a natural square root, namely a holomorphic line bundle $\mathbf{K}^{-1/2}$ whose square is isomorphic to the anticanonical bundle.

Definition 1.1. The fundamental line bundle on a twistor space is the holomorphic line bundle $\mathbf{K}^{-\frac{1}{2}}$. The corresponding linear system is called the fundamental system.

By construction, the fundamental line bundle is restricted to be the degree 2 line bundle on any fiber of the twistor fibration [2]. This basic topological observation will be very useful in our subsequent investigation.

On the twistor space, there is another very important structure, namely an antiholomorphic involution. We shall consider reality in terms of this *real structure.* For example, the fundamental bundle is real, so is the fundamental system. The fibers of the twistor fibration are also real. They are the *real twistor lines.*

In order to describe the twistor space, we shall use the associated map of the fundamental system as in [10], [18]. The next two lemmata are already proved in [10] and [18] implicitly.

Lemma 1.2 [10, Proposition (4.3)]. *Suppose that* $|V|$ *is a real linear subsystem of the fundamental system such that* $\dim|V| \geq 3$. *Then the system* $|V|$ *has no fixed components and a generic element in* $|V|$ *is non-singular irreducible.*

Proof. If the system $|V|$ were to have fixed component, then by [18, Lemma 2.1], the intersection number of the fixed component with a real twistor line is positive. As the system is real, this intersection number is at least two. On the other hand, the intersection number of the fundamental divisor with a real twistor line is equal to 2. Therefore, the existence of fixed component of $|V|$ implies that any movable part of $|V|$ has non-positive intersection with a real twistor line. This is a contradiction to [18, Lemma 2.1].

As the base locus of the system $|V|$ has dimension at most 1, one can follow the argument of [10, Proposition 4.3(iii)] to conclude that a generic real twistor line does not intersect the base locus of the linear system and subsequently conclude that the associated map of $|V|$ is a regular map in a neighborhood of a real twistor line. Checking the degree of the fundamental line bundle on the twistor line, we deduce that the image of a generic real line is a normal curve of degree not greater than 2. If the dimension of the system $|V|$ is at least 3 as assumed, then the image of the twistor space via the associated map of the system $|V|$ cannot be one-dimensional for otherwise, the image would have been contained in a plane that contains the image of a real twistor line.

As the image of the twistor space is at least 2-dimensional and the system $|V|$ has no fixed component, the first Bertini's theorem [1] implies that a generic element in $|V|$ is irreducible.

To prove the nonsingularity, we recall that if s is a real section of the fundamental line bundle whose divisor S is irreducible, then S is singular at a point p if and only if the section s and its derivative vanish along the real twistor line L containing p [10, Proposition 4.3(iii)].

If all elements in the vector space V have this twistor line L as singularity, V is a vector subspace of $H^0(Z, \mathcal{I}^2 \otimes \mathcal{K}^{-\frac{1}{2}})$ where $\mathcal{I}$ is the ideal sheaf of the twistor line L in the twistor space. However, the space $H^0(Z, \mathcal{I}^2 \otimes \mathcal{K}^{-\frac{1}{2}})$ is at most 3-dimensional [10]. Therefore, when the linear dimension of V is as large as 4, there is at least one member of the system that is nonsingular along L. Now, one can argue as Hitchin did by applying the second Bertini's theorem [1] to conclude that a generic real element in the system $|V|$ is nonsingular.

Lemma 1.3 [18, Lemma 2.5]. *Suppose that S is a real nonsingular irreducible element of the fundamental system on the twistor space of $\tau\mathbf{P}^2$. Then S is the blow-up of a rational ruled surface 2τ-times. The generic fiber of the ruling as a divisor on the surface S is linearly equivalent to a real twistor line contained in S.*

Lemma 1.4. *If the fundamental system has dimension at least 4, the base locus has dimension at most zero.*

Proof. After Lemma (1.2), we only need to prove that the base locus of the fundamental system cannot contain any curves.

Now suppose contrary to the lemma that the base locus of the fundamental system contains a curve C. Let L be any real twistor line through C. By the reality of the base locus of the fundamental system, L passes through at least two points in the base locus, namely a point in the intersection $L \cap C$ and the conjugate point. Let $|V|$ be the subsystem of the fundamental system passing through a point on L that is not in the intersection with C and $\overline{C}$. This system has dimension at least three. However, this system can also be considered as the subsystem containing L because $-\frac{1}{2}KL = 2$, and every element in $|V|$ passes through at least three points on L. In particular, the system $|V|$ is real. Now a generic real element S can be described by the last two lemmata.

By the adjunction formula, the restriction of the fundamental line bundle on the surface S is holomorphically isomorphic to the anticanonical bundle of the surface, we have the following exact sequence on the twistor space:

$$0 \to \mathcal{O} \to \mathcal{K}^{-\frac{1}{2}} \to \mathcal{K}_S^{-1} \to 0.$$

By the Ward correspondence [9], the Hodge number $h^{0,1}(Z)$ of the twistor space is equal to the first betti number of the 4-manifold X. As a connected-sum of complex projective planes is simply connected, the above exact sequence of sheaves induces an exact sequence of 0th cohomology:

$$(1.5) \qquad 0 \to H^0(Z, \mathcal{O}) \to H^0(Z, \mathcal{K}^{-\frac{1}{2}}) \to H^0(S, \mathcal{K}_S^{-1}) \to 0.$$

In particular, the base locus of the fundamental system of the twistor space is precisely the base locus of the anticanonical system of the surface S. By assumption, the fixed component of the anticanonical system of S, $|-K_S|$, contains at least the curve C and its conjugate. It follows that any element A in the anticanonical system is a sum of two parts, namely the fixed part F and a movable part E. Since (1.5) is exact and the dimension of the fundamental system is at least 4, the effective divisor E on the surface is moving in a family of at least 3 dimensions. As there is a real twistor line L contained in S such that it is linearly equivalent to a generic fiber of the blow-up of a ruled surface, the intersection number on S, EL, is strictly positive. By the reality of S, E and L, $EL \geq 2$. Since $-K_S L = -\frac{1}{2}K_{|S}L = 2$, then $FL = 0$. Yet, by the definition of the subsystem $|V|$, the curve C intersects the real twistor line L on the surface S. Therefore, L is a component of C. This is impossible because L is in the pencil of a generic fiber of the blow-up of a rational ruled surface and it cannot be a base locus of the anticanonical system of the surface. q.e.d.

Other than the fundamental line bundle, there are other natural holomorphic objects associated to the twistor space of a connected-sum of complex projective planes as follows: with respect to the intersection form of the 4-manifold X, there is an orthonormal basis of $H^2(X, \mathbf{Z})$:

$\{\alpha_j : j = 1, 2, \cdots, \tau\}$. Such basis is not unique as one can always replace α_j by $-\alpha_j$. For any choice of ordered orthonormal basis, when σ_j is an odd integer, the mod-2 reduction of $\sum \sigma_j \alpha_j$ is the second Stiefel-Whitney class of the manifold X, $w_2(X)$. On the other hand, $w_2(X)$ is equal to the second Stiefel-Whitney class of the fundamental line bundle, which is the mod-4 reduction of the first Chern class c_1 of the twistor space [10]. Therefore, the cohomology class

$$\tfrac{1}{4}c_1 + \tfrac{1}{2}\sum \sigma_j \alpha_j$$

is integral.

Note that $h^1(Z, \mathscr{O}) = 0$ because X is simply connected and that $h^2(Z, \mathscr{O}) = h^1(Z, \mathscr{K}) = 0$ due to the positivity of the scalar curvature [9]. Corresponding to the integral cohomology class $\frac{1}{4}c_1 + \frac{1}{2}\sum \sigma_j \alpha_j$, with σ_j odd, there is a unique holomorphic line bundle, $\mathbf{D}_{\sigma_1 \cdots \sigma_\tau}$. We shall use $\mathbf{D}$ or $\mathbf{D}_0$ to represent $\mathbf{D}_{1 \cdots 1}$, $\overline{\mathbf{D}}$ or $\overline{\mathbf{D}}_0$ to represent $\mathbf{D}_{-1 \cdots -1}$, $\mathbf{D}_j$ to represent $\mathbf{D}_{1 \cdots -1 \cdots 1}$, where -1 is at the jth slot and $\overline{\mathbf{D}}_j$ to represent $\mathbf{D}_{-1 \cdots 1 \cdots -1}$, where 1 is at the jth slot. $\overline{D}$ denotes the conjugation of D with respect to the real structure on the twistor space.

Definition 1.6. For any choice of orthonormal basis of the second integral cohomology on the 4-manifold X, the bundles $\mathbf{D}_j$ and $\overline{\mathbf{D}}_j$, $j = 0, 1, \cdots, \tau$, are called elementary bundles. If an elementary bundle has a meromorphic section, the divisor of the meromorphic section is called an elementary divisor.

In the rest of this paper, we shall rely heavily on the following isomorphism of line bundles:

$$\mathbf{D}_j \overline{\mathbf{D}}_j \cong \mathbf{K}^{-\frac{1}{2}}, \tag{1.7}$$

and the corresponding linear equivalence of divisors. The validity of this isomorphism is due to the fact that on our twistor spaces, holomorphic line bundles are uniquely determined by their first Chern classes.

In order to compute the intersection numbers between elementary divisors, we observe that

$$c_1 \alpha_j^2 = -4, \quad c_1^2 \alpha_j = 0, \quad c_2 \alpha_j = 0. \tag{1.8}$$

These formulas will also be needed when we compute the Euler characteristics of line bundles related to the elementary bundles.

In terms of cohomology of the twistor space, the Chern classes of the fundamental bundle and the elementary bundles span the entire second cohomology space of the twistor space. We shall demonstrate that they are also the essential data to describe the structure of a twistor space. Since the understanding of the algebraic structure of the elementary divisors, whenever it is effective, is the heart of this paper, we shall prove the following lemma although its contents can be traced to [18]:

Lemma 1.9. *If the line bundle* $\mathbf{D}_{\sigma_1\cdots\sigma_\tau}$ *has an effective divisor* D, *when* σ_j *'s are odd, then*

(1) $\sigma_j^2 = 1$; *i.e.* D *is an elementary divisor*;

(2) D *is a nonsingular irreducible surface*;

(3) *the surface* D *intersects its conjugate along a unique real twistor line with multiplicity* 1.

Proof. As

$$LD = \int_L \tfrac{1}{4}c_1 + \tfrac{1}{2}\sigma_j\alpha_j = \int_L \tfrac{1}{4}c_1 = 1,$$

according to [18, Lemma 2.1], D is irreducible.

Applying the technique of [10, Proposition 2.3], one can prove that if a holomorphic section of an elementary bundle vanishes to order 2 at a point, it vanishes to all order along the real twistor line through the singular point. By analyticity, this section vanishes identically on the twistor space. In particular, D is nonsingular as claimed in (2).

As $LD = 1$, if D does not contain any real twistor line, the twistor fibration is restricted to be a diffeomorphism from the compact complex surface D onto the 4-manifold X. Moreover, with respect to the natural orientation induced by the complex structure on the surface D, the diffeomorphism is orientation reversing. Let c_1' and c_2' be the Chern classes of the surface D. When ξ and τ are the Euler number and signature of the manifold X, the existence of this orientation reversing diffeomorphism between D and X implies that

$$\xi = c_2', \qquad -\tau = \tfrac{1}{3}(c_1'^2 + 2c_2').$$

It follows that

$$\tfrac{1}{3}(c_1'^2 + c_2') = \xi - \tau.$$

As the manifold X is homeomorphic to a connected-sum of complex projective planes, $\xi - \tau = 2$ and hence $\frac{1}{12}(c_1'^2 + c_2') = \frac{1}{2}$. However, by the Atiyah-Singer index theorem, the Euler characteristic of the holomorphic tangent bundle of the surface D is equal to $\frac{1}{12}(c_1'^2 + c_2')$. In particular, it should have been an integer. This contradiction shows that the surface D must contain at least one real twistor line.

Let L be a real twistor line contained in D. As it is real, it is contained in $D \cap \overline{D}$. Note that as D and $\overline{D}$ always intersect at conjugate pair of points, their intersection is a finite union of real twistor lines. To see that this intersection is a single twistor line with multiplicity 1, we apply the adjunction formula on the twistor space and then find that

$$K_D L = \left(-\tfrac{3}{4}c_1 + \tfrac{1}{2}\sum \sigma_j \alpha_j\right) L = -\tfrac{3}{4}c_1 L = -3.$$

As L is a nonsingular irreducible rational curve, the adjunction formula on the surface D shows that $L^2 = 1$. On the other hand, applying (1.8), we have

$$\begin{aligned}(\overline{D}_{|D})^2 &= \left(\tfrac{1}{4}c_1 + \tfrac{1}{2}\sum \sigma_j\alpha_j\right)\left(\tfrac{1}{4}c_1 - \tfrac{1}{2}\sum \sigma_j\alpha_j\right)^2 \\ &= \tfrac{1}{64}c_1^3 - \tfrac{1}{16}\sum c_1\alpha_j^2 = 1.\end{aligned}$$

Therefore, if $\overline{D}_{|D} = \sum n_i L_i$, where the L_i's are real twistor lines, then as real twistor lines are mutually disjoint and have self-intersection number 1 on the surface D, the last equation shows that $\sum n_i^2 = 1$. Therefore, D intersects $\overline{D}$ along a unique real twistor line with multiplicity 1. Hence (3) is proved.

Finally, since the self-intersection number of the unique real twistor line on the surface D is equal to 1, the twistor fibration shows that D is diffeomorphic to the connected-sum $\mathbf{P}^2\#X'$, where X' is the manifold X with opposite orientation. As X is homeomorphic to $\tau\mathbf{P}^2$, $K_D^2 = c_1^2(D) = 9-\tau$.

On the other hand, $K_D = (K+D)_{|D} = -\frac{3}{4}c_1 + \frac{1}{2}\sum\sigma_j\alpha_j$,

$$\begin{aligned}K_D^2 &= \left(\tfrac{1}{4}c_1 + \tfrac{1}{2}\sum \sigma_j\alpha_j\right)\left(-\tfrac{3}{4}c_1 + \tfrac{1}{2}\sum \sigma_j\alpha_j\right)^2 \\ &= 9 - \tfrac{9}{4}\tau + \tfrac{5}{4}\sum \sigma_j^2.\end{aligned}$$

Therefore, $\sum\sigma_j^2 = \tau$ and hence $\sigma_j^2 = 1$. Then (1) is proved. q.e.d.

Lemma 1.10. *Suppose that D is an effective elementary divisor. Let L be the unique real twistor line on D. Then the following hold:*

(1) *The associated map ψ_L of the complete linear system $|L|$ on D exhibits D as a blow-up of $\mathbf{P}^2$ τ-times such that L is linearly equivalent to the proper transform of the hyperplane class H on $\mathbf{P}^2$.*

(2) *After appropriate reordering of the basis $\{\alpha_j : j = 1, \cdots, \tau\}$ and sign changing, we can assume that D is an effective divisor of the bundle $\mathbf{D}_0$. Let H be the hyperplane class and E_i be the exceptional divisor of*

the blow-up map ψ_L. *Then*

$$\overline{D}_{|D} = H\,, \qquad D_{|D} = H - \sum E_i\,,$$
$$-\frac{1}{2}K_{|D} = 2H - \sum E_i\,, \qquad D_{\sigma_1 \cdots \sigma_{\tau|D}} = H - \frac{1}{2}\sum(1+\sigma_j)E_j.$$

Proof. Part (1) is an elementary observation in algebraic geometry. It can be proved as follows: Note that $h^{0,1}(Z) = 0$ due to the Ward correspondence and the simple connectivity of X and that $h^2(Z, \mathscr{D}^{-1}) = h^1(Z, \mathscr{K}\mathscr{D}) = 0$ due to the Serre duality and Hitchin's vanishing theorem [9]. Then the exact sequence

$$0 \to \mathscr{D}^{-1} \to \mathscr{O} \to \mathscr{O}_D \to 0$$

implies that $h^1(D, \mathscr{O}_D) = 0$. As the self-intersection number of L on the surface D is equal to 1 as proved in the last lemma, the following sequence is exact:

$$0 \to \mathscr{O}_D \to \mathscr{O}_D(\mathbf{L}) \to \mathscr{O}_L(1) \to 0.$$

It induces the following exact sequence:

$$(1.11) \qquad 0 \to H^0(D, \mathscr{O}_D) \to H^0(D, \mathscr{O}_D(\mathbf{L})) \to H^0(L, \mathscr{O}_L(1)) \to 0$$

because $h^1(D, \mathscr{O}_D) = 0$. It follows that the base locus of $|L|$ on D is contained in the base locus of the complete linear system of degree 1 divisor on L and hence is empty. Therefore, the associated map ψ_L is holomorphic. The exact sequence (1.11) also implies that the restriction of ψ_L onto any irreducible element in $|L|$ is an embedding to a line. Counting the degree of the image, we see that the associated map has to be surjective. Then the same degree would show that the map is generically one-to-one. Therefore, ψ_L blows down D to $\mathbf{P}^2$ as claimed. By construction, the real twistor line is linear equivalent to the proper transform of a generic line on $\mathbf{P}^2$. Part (1) is proved.

To prove part (2), recall that $\overline{D}$ intersects D along the real twistor line L with multiplicity 1 and the real twistor line is linearly equivalent to the hyperplane class; then on the surface D, $\overline{D}_{|D} = H$. By the adjunction formula and (1.7),

$$-K_D = (-K - D)_{|D} = (2D + 2\overline{D} - D)_{|D} = (D + 2\overline{D})_{|D} = D_{|D} + 2H.$$

While $-K_D = 3H - \sum E_i$, $D_{|D} = H - \sum E_i$. It also follows that $-\frac{1}{2}K_{|D} = 2H - \sum E_j$.

As $1 = \overline{D}_j L = \overline{D}_{j|D} H$, $\overline{D}_{j|D} = H - \sum n_i E_i$. Since $1 - \sum n_i^2 = (\overline{D}_{j|D})^2 = 0$, $\overline{D}_{j|D} = H - E_{j_0}$, for some j_0. Moreover, for $i \neq j$, $\overline{D}_{i|D}\overline{D}_{j|D} = 1$. Up to permutation, $\overline{D}_{j|D} = H - E_j$. As

$$D_{j|D} + \overline{D}_{j|D} = D_{|D} + \overline{D}_{|D} = 2H - \sum E_i,$$

then $D_{j|D} = H - \sum_{i \neq j} E_i$.

In general, $D_{\sigma_1 \cdots \sigma_\tau |D} H = 1$; then $D_{\sigma_1 \cdots \sigma_\tau |D} = H - \sum n_i E_i$ for some n_i. As

$$D_{\sigma_1 \cdots \sigma_\tau |D} \overline{D}_{j|D} = 1 - \tfrac{1}{2}(1 + \sigma_j),$$

$n_i = \frac{1}{2}(1 + \sigma_j)$.

2. The generic twistor spaces over $3\mathbf{P}^2$

On any twistor space associated to a self-dual metric of positive scalar curvature on the connected-sum of three copies of complex projective plane, $3\mathbf{P}^2$, one can apply the Riemann-Roch formula to show that the Euler characteristic of any elementary line bundle is equal to 1. Since the tensor product of the canonical bundle of the twistor space with the dual of any elementary line bundle has degree -5 on any real twistor line, by the Serre duality, $h^3(Z, \mathbf{D}) = h^0(Z, \mathbf{KD}) = 0$ for any elementary line bundle $\mathbf{D}$. Similarly, the Serre duality and Hitchin's vanishing theorem [9] implies that $h^2(Z, \mathbf{D}) = 0$. Therefore,

$$h^0(Z, \mathbf{D}) - h^1(Z, \mathbf{D}) = 1.$$

In particular, all elementary line bundles have an effective divisor. According to Lemma (1.10), the elementary divisors are the blow-ups of $\mathbf{P}^2$ three times.

Note that if $\mathbf{D}$ is an elementary line bundle with $h^1(Z, \mathbf{D}) \geq 1$, the corresponding linear system has dimension at least one and its restriction onto any effective elementary divisor D has effective divisor. In other words, the restriction $D_{|D}$ is effective. By Lemma (1.10), D is a blow-up of $\mathbf{P}^2$ at three collinear points. As far as the blowing-up of $\mathbf{P}^2$ is concerned, a generic blow-up of $\mathbf{P}^2$ is to blow up three distinct noncollinear points. On the other hand, it is proved in [14] and [15] that given a smooth 1-parameter family M_t of surfaces obtained from $\mathbf{P}^2$ by blowing up distinct points, there is a smooth family of twistor spaces Z_t containing effective divisors D_t such that $D_t \cong M_t$. Therefore, we consider any twistor spaces associated to $3\mathbf{P}^2$ such that *all* elementary divisors are blow-ups of $\mathbf{P}^2$ at three *generic* noncollinear points. This section is devoted to giving an algebraic description of such generic twistor spaces. Equivalently, we shall assume that $h^1(Z, \mathscr{D}) = 0$.

Theorem 2.1. *The twistor space associated to a generic self-dual conformal class containing a metric of positive scalar curvature on* $3\mathbf{P}^2$ *is a small resolution of the double covering of* $\mathbf{P}^3$ *branched over a quartic with thirteen ordinary double points. Exactly one of these points is real. In homogeneous coordinates* $\{z_0, z_1, z_2, z_3\}$ *on* $\mathbf{P}^3$, *the equation of the quartic is*

$$B(z) = z_0 z_1 z_2 z_3 - Q^2(z),$$

where Q *is a real positive definite quadric.*

The main point of the proof is to realize that, in the generic case, the associated map of the fundamental system is a double covering map and that the elementary divisors will help to determine the singularities. To prove the first claim, we apply the Riemann-Roch formula and Hitchin's vanishing theorem [9] to find that

$$h^0(Z, \mathscr{K}^{-\frac{1}{2}}) - h^1(Z, \mathscr{K}^{-\frac{1}{2}}) = 4.$$

In fact, $h^1(Z, \mathscr{K}^{-\frac{1}{2}})$ is equal to zero. If it were not equal to zero, the dimension of the fundamental system would be at least 4. In particular, according to Lemma (1.2) and (1.3), a generic real element S of the fundamental system is a nonsingular irreducible rational surface. According to Lemma (1.4), the dimension of the base locus of the fundamental system is at most zero. Since the restriction of the fundamental line bundle onto the surface S is precisely the anticanonical bundle $\mathbf{K}_S^{-1}$ of the surface S, if C is any irreducible curve on the surface S, the intersection number on S, $-K_S C$, is nonnegative. Therefore, S is a degenerate del Pezzo surface in the sense of Demazure [6]. In particular, $h^1(S, \mathscr{K}^{-1})$ is equal to zero. On the other hand, $h^1(Z, \mathscr{O})$ and $h^2(Z, \mathscr{O})$ are both equal to zero; the following exact sequence on the twistor space:

$$0 \to \mathscr{O} \to \mathscr{K}^{-\frac{1}{2}} \to \mathscr{K}_S^{-1} \to 0$$

would imply a contradiction that $h^1(Z, \mathscr{K}^{-\frac{1}{2}}) = 0$. Therefore, the fundamental system has dimension 3 and hence the associated map is a map into $\mathbf{P}^3$.

Proposition 2.2. *The base locus of the fundamental system of a generic twistor space is an empty set.*

Proof. Let D be any effective elementary divisor. According to Lemma (1.9), D is a nonsingular irreducible surface. Due to the isomorphism (1.7), we have the following exact sequence:

$$0 \to \overline{\mathscr{D}} \to \mathscr{K}^{-\frac{1}{2}} \to \mathscr{K}_{|D}^{-\frac{1}{2}} \to 0. \tag{2.3}$$

When $h^1(Z, \mathscr{D})$ is equal to zero as we assume, $h^1(Z, \overline{D}) = 0$ by the reality. Then this exact sequence induces an exact sequence of 0th cohomology groups. In particular, the fundamental system has no base point if its restriction on D is base point free.

Due to the isomorphism (1.7) and Lemma (1.9), one has the following exact sequence on D:

$$0 \to \mathscr{D}_{|D} \to \mathscr{K}_{|D}^{-\frac{1}{2}} \to \mathscr{K}_{|L}^{-\frac{1}{2}} \to 0. \tag{2.4}$$

On the other hand, the exact sequence on the twistor space

$$0 \to \mathscr{O} \to \mathscr{D} \to \mathscr{D}_{|D} \to 0$$

implies that

$$h^0(D, \mathscr{D}_{|D}) = h^1(D, \mathscr{D}_{|D}) = 0.$$

Then the induced long exact sequence of (2.4) implies that the restriction map induces a natural isomorphism:

$$H^0(D, \mathscr{K}_{|D}^{-\frac{1}{2}}) \cong H^0(L, \mathscr{K}_{|L}^{-\frac{1}{2}}).$$

As the restriction of the fundamental bundle onto any real twistor line is isomorphic to the degree 2 bundle on a rational curve, it is base point free and hence the restriction of the fundamental system on D is also base point free. q.e.d.

Since the fundamental system is free, the associated map $\mathbf{\Phi}$ is a holomorphic map from the twistor space into $\mathbf{P}^3$.

Lemma 2.5. *The associated map* $\mathbf{\Phi}$ *of the fundamental system is a double covering of* $\mathbf{P}^3$ *branched along a quartic.*

Proof. Let E be the intersection of two generic real elements in the fundamental system. It can be treated as an effective divisor of the anticanonical system of one of the two elements, say S. Since the fundamental system is base point free, so is the anticanonical system of the surface S. Therefore, E as a generic element in $|-K_S|$ is a nonsingular elliptic curve. The exact sequence on S

$$0 \to \mathscr{O}_S \to \mathscr{K}_S^{-1} \to \mathscr{O}_E(\mathscr{K}_S^{-1}) \to 0$$

induces an exact sequence of 0th cohomology because S is a rational surface. Therefore, the restriction of $\mathbf{\Phi}$ onto E is the associated map of the complete linear system of $|-K_{S|E}|$, which is $|-\frac{1}{2}K_{|E}|$. Since the degree of $-\frac{1}{2}K$ on E is equal to

$$(-\tfrac{1}{2}K)^3 = \tfrac{1}{8}c_1^3 = 2,$$

the map $\mathbf{\Phi}$ exhibits E as a double covering from an elliptic curve onto a line in $\mathbf{P}^3$. Therefore, $\mathbf{\Phi}$ is a double covering map.

Since the ramification locus is the zeros of the determinant of the Jacobian, it is an effective divisor of the bundle $\mathrm{Hom}(\mathbf{\Phi}^*\mathbf{H}^{-4}, \mathbf{K})$. As $\mathbf{\Phi}^*\mathbf{H} = \mathbf{K}^{-\frac{1}{2}}$, the ramification locus is a divisor of $\mathbf{K}^{-1}$ and the branched locus in $\mathbf{P}^3$ is a quartic. This kind of variety was referred to as a double solid [5]. q.e.d.

Recall that each elementary bundle on the twistor space of $3\mathbf{P}^2$ has an effective divisor. To find the equation and singularity of the quartic, we examine the elementary bundles $\mathbf{D}_j$.

Proposition 2.6. *Each effective elementary divisor D_j is a blow-up of $\mathbf{P}^2$ at three points in general position.*

Proof. It suffices to prove this proposition for D. According to Lemma (1.10), if E_1, E_2 and E_3 represent the exceptional divisors of blowing-up, $D_{|D}$ is linearly equivalent to $H - E_1 - E_2 - E_3$. As $D_{|D}$ is not effective, the three points of blowing-up are not collinear.

To complete the proof, we have to show that no two points are infinitely near. As $-\frac{1}{2}K_{|D}(E_3 - E_1 - E_2) = -1$ and the fundamental system has no base points, no two points can be infinitely near the third point. Suppose that the third point of the blowing-up is infinitely near the second point of the blowing-up; then $E_2 - E_3$ is effective and the complete linear system of $H - E_3$ has $E_2 - E_3$ in its fixed component. In fact, we have either $D_{3|D}$ decomposed into $(H - E_2) + (E_2 - E_3)$ or into $(H - E_1) + (E_1 - E_2) + (E_2 - E_3)$. In both cases, these divisors on D can be considered as the intersection of D and $\overline{D}_3$. In this intersection, there are two irreducible nonsingular rational curves intersecting transversely at one point, say p. As divisors in the twistor space, D and $\overline{D}_3$ are both irreducible nonsingular. The above intersection configuration is possible only if D and $\overline{D}_3$ have at least second order of contact at the point p.

On the other hand, the associated map sends both D and $\overline{D}_3$ to a plane in $\mathbf{P}^3$ because the fundamental system is restricted onto D, and also onto $\overline{D}_3$, to be the complete system of conics through the three noncollinear points of blowing-up. If D and $\overline{D}_3$ were to have at least second order of contact at one point, their images through the associated map would have been an identical plane. It would imply, in turn, that $D = \overline{D}_3$. Checking their intersection numbers, one can see that this is a contradiction. q.e.d.

After the last proposition, it is clear that $D \cap D_j$ and $D \cap \overline{D}_j$ are irreducible rational curves. In particular, the associated map sends $D \cap D_j$, $j = 1, 2, 3$, to three distinct lines on the plane $P_0 := \Phi(D)$. Therefore, the planes $P_j := \Phi(D_j) = \Phi(\overline{D}_j)$ are all distinct. We choose a homogeneous coordinate z on $\mathbf{P}^3$ such that, for $j = 0, 1, 2, 3$,

$$P_j = \{z \in \mathbf{P}^3 : z_j = 0\},$$

and we shall consider the union of these four planes as a tetrahedron, denoted by T. On each face of this tetrahedron, there is a distinguished conic, namely the image of the real twistor line L_j on the divisor D_j. We shall use the same symbol L_j to denote these conics in $\mathbf{P}^3$.

On each such conic, there are three pairs of distinguished points, namely the intersection of this conic and the three edges of the tetrahedron on the given face. Each pair of these points on an edge is the intersection of the conics on the pair of faces sharing the given edge. The tetrahedron has six edges and hence six such pairs of points.

On the face P_0, there are the images of $D \cap D_j$. As $-\frac{1}{2}KDD_j = 0$, the image of $D \cap D_j$ is a point. These are three of the six distinguished points on the conic L_0. The remaining three distinguished points are the images of the conjugate curves. All the other distinguished points on the edges of the tetrahedron can be described in a similar fashion. The essential observation is that they are the images of some irreducible nonsingular rational curves. As the associated map is a double covering map, it is possible only if Φ is a small resolution of the double covering of $\mathbf{P}^3$ branched over a quartic, B, with at least these six pairs of points as singularities.

We claim that the quartic B is given by

$$B = \{z \in \mathbf{P}^3 : B(z) \equiv z_0 z_1 z_2 z_3 - tQ^2(z) = 0\}, \tag{2.7}$$

where t is a nonzero real number and $Q(z)$ is a quadratic homogeneous polynomial such that the corresponding quadric is a real quadric surface containing the four conics L_j's.

First of all, let us show that there is a quadric Q containing the four conics: Let

$$L_0 \cap L_1 = \{x_1, x_2\}, \quad L_0 \cap L_2 = \{x_3, x_4\}, \quad L_0 \cap L_3 = \{x_5, x_6\},$$
$$L_1 \cap L_2 = \{x_7, x_8\}, \quad L_1 \cap L_3 = \{x_9, x_{10}\}, \quad L_2 \cap L_3 = \{x_{11}, x_{12}\}.$$

Within the 4-parameter family of quadrics containing L_0, there is at least a 1-parameter family Q_l, $l \in \mathbf{C}$, containing x_7, x_8 and x_9. The plane of these three points interesects Q_l along a conic. This conic also contains x_1 and x_2 because these two points are also on the plane of x_7, x_8 and x_9. Yet five points in general position on a plane uniquely determine a conic. Therefore, this conic has to be the L_1. Within this 1-parameter family of quadrics containing L_0 and L_1, there is at least one containing the point x_{11}. But the plane of x_{11}, x_3, x_4 intersects this quadric along a conic containing x_{11}, x_3, x_4 and x_7, x_8. Hence this quadric contains L_2 as well. Now, this quadric contains x_{12}, x_5, x_6, x_9 and x_{10}. Therefore, it also contains L_3. Knowing that there is at least one quadric containing L_0, L_1, L_2 and L_3, we can choose a real one because the L_j's are real.

Let $T_t := \{z \in \mathbf{P}^3 : B(z) + tQ^2(z) = 0\}$. This is a 1-parameter family of quartics. The intersection of this quartic and a P_j contains at least the double conic L_j because both B and Q^2 vanish on L_j with multiplicity 2. However, for a general t, T_t contains some points other than those on L_j. It is possible only if T_t contains the plane P_j. Therefore, the equation of B is given as in (2.7). Now the t has to be real because both Q and B are real.

Though the ramification locus Y of the covering map $\mathbf{\Phi}$ is a real space in the twistor space without real points, the branch locus B in $\mathbf{P}^3$ contains real points. To prove this claim, let L be a real twistor line whose image passes through a vertex of the tetrahedron. As the vertex is a real point, it must be the image of a conjugate pair of points. In particular, the map $\mathbf{\Phi}$ cannot be an embedding on L. In fact, it has to be a double covering of the line $\mathbf{\Phi}(L)$ branched over a conjugate pair of points. Therefore, the real line $\mathbf{\Phi}(L)$ intersects the quartic at least at two distinct points. Yet, from the equation (2.7) of B, a direct algebraic computation shows that a real line in $\mathbf{P}^3$ through a vertex of the tetrahedron can intersect B at a conjugate pair of points with multiplicity 2 only when it is an edge of the tetrahedron. Therefore, $\mathbf{\Phi}(L)$ has to intersect B at least at three points. However, $\mathbf{\Phi}(L)$ cannot intersect B at four distinct points, i.e. two conjugate pairs of points, for otherwise, the real twistor line L would have intersected the ramification locus, which is a divisor of $\mathbf{K}^{-1}$, at six points. Therefore, the real line $\mathbf{\Phi}(L)$ intersects B at three points. Then one of them must be real.

Proposition 2.8. *There is one and only one real point on* B.

Proof. Since the inverse images of any real points have to contain at least one conjugate pair of points, a real point on the branch locus B cannot be smooth. Fix any real point, say u, on B. This point is not on the tetrahedron T because the intersection of T and B is a real conic with no real point. Let z, $\bar{z}$ be a conjugate pair of points in Z such that their images are u.

Let $|-\frac{1}{2}K|_u$ be the system of fundamental divisors containing z and $\bar{z}$. In $\mathbf{P}^3$, this is the real system of hyperplanes with u as its base locus. Note that due to Lemma (1.9) the only reducible elements of the fundamental system are $D_j + \overline{D}_j$, $j = 0, 1, 2$, or 3. As u is not on the tetrahedron, every element in $|-\frac{1}{2}K|_u$ is irreducible. After an argument of Hitchin [10], an irreducible real element in such a system can have a singularity only if it has an entire real twistor line as singularity. Yet Bertini's second theorem asserts that a generic element in $|-\frac{1}{2}K|_u$ can be singular only along the inverse image of u. However, $\mathbf{\Phi}^{-1}(u)$ cannot contain any real twistor line because the image of any real twistor line has to intersect all faces of the tetrahedron, while the point u is not on the tetrahedron. Therefore, a generic element S in $|-\frac{1}{2}K|_u$ is an irreducible nonsingular surface.

As the base locus of the anticanonical system of S is the base locus of the fundamental system of the twistor space, it is an empty set. In particular, $-K_S C \geq 0$ for any curve C on the surface S. Therefore, S is the blowing-up of $\mathbf{P}^2$ at seven points in almost general position. Moreover, the seven points cannot be in general position, for otherwise the anticanonical map on S, i.e. the restriction of $\mathbf{\Phi}$ on S, would have been a double covering of $\mathbf{P}^2$ branched over a nonsingular quartic. Since the point u is singular on B, a generic S must contain (-2)-curves. Moreover, u is the image of a (-2)-curve, say C, on S so that $\mathbf{\Phi}(S) \cap B$ contains u as an ordinary double point.

Note that if a real twistor passes through the (-2)-curve C on the surface S, its image is a real line in $\mathbf{P}^3$ passing through u. Then this line intersects the real plane P_0 at one real point. Since the associated map restricted onto the elementary divisor D_0 is a blowing-down map, the image of different real twistor lines through C intersects P_0 at different real points. Therefore, real twistor lines through C are parametrized by the real part of P_0, i.e. a copy of $\mathbf{RP}^2$. Meanwhile, the curve C is diffeomorphic to S^2 as it is a smooth rational curve. Through every point of C, there is a unique real twistor line. Therefore, C must be real and any real twistor line through C passes through C at a conjugate pair of points. Moreover, the real point u on B is the base locus of the system of all twistor lines whose image via $\mathbf{\Phi}$ is a line. In particular, the point u is the unique real point on B as claimed. q.e.d.

Suppose that p and $\bar{p}$ are a conjugate pair of singular points different from the twelve known nonreal singularities, and let ℓ be the line joining p and $\bar{p}$. Since ℓ is a real line and B contains only one real point, this line intersects B nowhere other at p and $\bar{p}$. In particular, it is disjoint from the thirteenth singular point u on B.

As ℓ is disjoint from u, it is not the image of a real twistor line. It follows that a generic element of the pencil, $|-\frac{1}{2}K|_\ell$, of elements containing $\Phi^{-1}(\ell)$ is irreducible nonsingular. The argument to prove this observation is similar to the one that we applied on $|-\frac{1}{2}K|_u$ in the proof of the last proposition. Let S be a generic real element. It is a blow-up of $\mathbf{P}^2$ seven times to a degenerate del Pezzo surface. The fundamental system on the twistor space restricted onto S has to be the anticanonical system of S. The associated map exhibits it as a double covering of $\mathbf{P}^2$ branched along a singular quartic with a conjugate pair of singular points. Therefore, S would contain a conjugate pair of (-2)-curves. As S intersects D_j along (-1)-curves, we can work out the configuration of (-1)-curves along with the real structure on S and then show that it is impossible to have only one conjugate pair of (-2)-curves. This technical computation on S will be demonstrated in the appendix. The conclusion is that the branch locus B has exactly thirteen singular points. One of them is real. As with the six conjugate pairs of singular points on B, u is also an ordinary double point. This is due to the fact that every real element in $|-\frac{1}{2}K|_u$ is a nonsingular irreducible element containing C. Therefore, each real plane containing u intersects the quartic B along an irreducible curve with u as an ordinary double point.

As u is a real point not on the tetrahedron, we can find a real coordinate such that u is the unit point, i.e. $u = [1, 1, 1, 1]$. As B contains u,

$$tQ^2(u) = 1.$$

Therefore, t is a positive number. Letting the equation of Q absorb the number t, we can assume that the equation of the quartic B is

$$B(z) = z_0 z_1 z_2 z_3 - Q^2(z). \tag{2.9}$$

As Q contains the real conics L_j without real points, Q is definite. If necessary, replacing Q by $-Q$, we can assume that Q is positive definite.

So far, we have finished the proof of the characterization Theorem (2.1) without discussing the existence. It is conceivable that one may apply the twistor programme as in [18] to construct a twistor space over $3\mathbf{P}^2$. An incomplete attempt was given in [20]. For a complete construction of generic twistor spaces over $3\mathbf{P}^2$, readers are referred to [7]. In the next paragraph, we shall merely demonstrate that an algebraic variety described in Theorem (2.1) does exist.

The quadric Q in (2.9) is not generic. In general, the quartic given by (2.9) has only twelve ordinary double points. They are the intersection points of four double conics on the quartic B given by the intersection of the quadric Q and the tetrahedron T. This type of quartic was studied by Kummer ([3], [11], [12]). When B is required to contain the point u as an ordinary double point, the coefficients of the quadric Q are subject to four independent linear conditions. As there are ten linear parameters in the choice of quadrics in $\mathbf{P}^3$, there is a 6-parameter family of quartics with the required singularity. In this 6-dimensional space, there is an open set parametrizing all those quartics with u as the sole real point. In fact, as long as the coefficients of the quadric Q are chosen so that the ellipsoid

$$Q = \{x \in \mathbf{R}^4 : Q(x) = 1\}$$

and the quartic

$$T = \{x \in \mathbf{R}^4 : x_0x_1x_2x_3 = 1\}$$

intersect only at $(1, 1, 1, 1)$ and $(-1, -1, -1, -1)$ and intersect tangentially at these points, then u is the sole real point on the quartic. Such an ellipsoid exists because we can choose the longest axis of the ellipsoid to be the line joining $(1, 1, 1, 1)$ and $(-1, -1, -1, -1)$.

3. A classification of LeBrun twistor spaces

In the last section, we described the twistor spaces of $3\mathbf{P}^2$ on which every elementary divisor is rigid in the sense that its complete linear system has a single element. In this section, we shall study the twistor spaces of $\tau\mathbf{P}^2$, $\tau \geq 3$, such that there is a pencil of effective elementary divisors. The following theorem completes the classification of twistor spaces of $3\mathbf{P}^2$. At the same time, it describes a very special family of twistor spaces on $\tau\mathbf{P}^2$ for any τ larger than 3.

Theorem 3.1. *Let Z be a twistor space associated to a self-dual metric of positive scalar curvature on $\tau\mathbf{P}^2$, $\tau \geq 3$. Suppose that there is an elementary line bundle such that its complete linear system of effective divisors is at least 1-dimensional. Then the metric is contained in LeBrun's self-dual conformal class. Equivalently, the twistor space is the LeBrun twistor space. In particular, the twistor space is Moishezon.*

The basic observation in the proof of this theorem is that when there is a pencil of effective elementary divisors, due to the isomorphism (1.7), the fundamental system is at least 3-dimensional. We shall use the associated map of the fundamental system to produce an algebraic description of the twistor space. It can be done because the twistor space is almost foliated by the pencil of effective elementary divisors and we can describe the associated map of the fundamental system on each elementary divisor.

When there is an elementary line bundle whose complete linear system of effective divisors is at least 1-dimensional, we choose an orthonormal basis of the second integral cohomology of the 4-manifold X such that $\dim|D| \geq \dim|D_j|$ for all j. Then $\dim|D| \geq 1$.

As $h^{0,1}(Z) = 0$, the exact sequence on the twistor space:

$$0 \to \mathscr{O} \to \mathscr{D} \to \mathscr{O}_D(\mathbf{D}) \to 0$$

induces an exact sequence of 0th cohomology:

$$(3.2) \qquad 0 \to H^0(Z, \mathscr{O}) \to H^0(Z, \mathscr{D}) \to H^0(D, \mathscr{O}_D(\mathbf{D})) \to 0,$$

where D is any element in the system $|D|$. According to Lemma (1.10), the system $|D_{|D}|$ on the surface D is the system of lines through all blow-up points. This system is nonempty only when it has a single element. Therefore, when $\dim|D| \geq 1$, then on the twistor space $\dim|D| = 1$ and on the surface D, we have $\dim|D_{|D}| = 0$.

Let C be the intersection of any two distinct elements, say D and D', in the pencil $|D|$. It can be considered as the sole effective divisor in the system $|D_{|D}|$. Referring to the description of the surface D as given in Lemma (1.10), we can conclude that the surface D is a blow-up of $\mathbf{P}^2$ τ-times on a line so that the curve C is the proper transform of the line through all points of blowing-up.

Proposition 3.3. *Suppose that* $\dim|D| \geq 1$; *then*

(1) $\dim|D| = 1$,

(2) $\dim|D_j| = 0$ *for all* $j = 1, \cdots, \tau$.

Proof. We have seen that (1) is basically due to the exactness of (3.2).

To prove (2), we recall that the Serre duality and the positivity of the scalar curvature imply [9] that

$$h^2(Z, \mathscr{D}_j\mathscr{D}^{-1}) = h^1(Z, \mathscr{K}\mathscr{D}_j^{-1}\mathscr{D}) = 0.$$

Also, in general, we have [10]

$$h^0(Z, \mathscr{D}_j\mathscr{D}^{-1}) = h^3(Z, \mathscr{D}_j\mathscr{D}^{-1}) = 0.$$

By the Riemann-Roch formula, one can check that

$$h^1(Z, \mathscr{D}_j\mathscr{D}^{-1}) = 0.$$

Therefore, the induced long exact sequence of

$$0 \to \mathscr{D}_j\mathscr{D}^{-1} \to \mathscr{D}_j \to \mathscr{D}_{j|D} \to 0$$

implies that there is a natural isomorphism:

$$H^0(Z, \mathscr{D}_j) \cong H^0(D, \mathscr{D}_{j|D}).$$

On the other hand, by Lemma (1.10),

$$D_{j|D} = \left(H - \sum E_i\right) + E_j .$$

Recall that the curve C is precisely $H - \sum E_i$; then D_j is effective and the system $|D_j|$ has precisely one element. q.e.d.

Proposition 3.4. *A generic element in the pencil $|D|$ is the blow-up of P^2 at τ distinct points on a line.*

Proof. It is enough to show that $E_i - E_j$ is not effective on D. If $E_i - E_j$ were effective, then on D

$$D_{i|D} = C + E_i = C + E_j + (E_i - E_j)$$

is effective. But $C + E_j$ is $D_{j|D}$. Therefore, D_i and D_j intersect along C as well as along E_j on any element in the pencil $|D|$.

On the other hand, as D_i and D_j are two fixed hypersurfaces of the compact twistor space, their intersection is a union of finitely many curves. Therefore, some of the E_j's obtained by the intersection of $D_i \cap D_j$ with different elements in the pencil $|D|$ are identical. Therefore, the pencil $|D|$ contains a base curve different from the curve C. This is a contradiction to our previous observation that the system $|D_{|D}|$ on the surface D has only one irreducible element, namely the curve C. q.e.d.

As $H^0(Z, \mathscr{D})$ and $H^0(Z, \overline{\mathscr{D}})$ are 2-dimensional, let $\{d, d'\}$ and $\{\overline{d}, \overline{d}'\}$ be their bases respectively. Due to the isomorphism (1.7), the dimension of the fundamental system is at least three. This system has base locus because when C is the base locus of the pencil $|D|$, its intersection number with the fundamental class is equal to

$$\begin{aligned}\frac{1}{2}c_1\left(\frac{1}{4}c_1 + \frac{1}{2}\sum \alpha_j\right)^2 &= \frac{1}{32}c_1^3 + \frac{1}{8}\sum c_1\alpha_j^2 \\ &= \frac{1}{2}(4-\tau) - \frac{1}{2}\tau = 2 - \tau \leq -1, \quad \text{when } \tau \geq 3.\end{aligned}$$

Then Lemma (1.4) shows that the dimension of the fundamental system is at most three. Therefore, the range of the associated map is $\mathbf{P}^3$, and the map can be written as

$$[d\overline{d}, d\overline{d}', d'\overline{d}, d'\overline{d}'].$$

The range is in a nonsingular real quadric, denoted by Q, in $\mathbf{P}^3$. Moreover, as the system $|D|$ has C as its only base locus, the fundamental system has no base points other than C and $\overline{C}$. Therefore, the associated map blows up along precisely C and $\overline{C}$.

To describe the twistor space as an object associated to the quadric, we blow up C and $\overline{C}$. The normal bundle $\mathscr{N}$ of C in the twistor space is given by the extension

$$0 \to \mathscr{N}_C^D \to \mathscr{N} \to \mathscr{N}_{D|C}^Z \to 0,$$

where $\mathscr{N}_C^D$ is the normal bundle of C in D. But $\mathscr{N}_C^D$ is precisely the restriction of the normal bundle of D in the twistor space onto C, i.e., $\mathscr{N}_{D|C}^Z$. Therefore the exceptional divisor of the blowing-up consists of two quadrics Q_C and $Q_{\overline{C}}$.

Let $\hat{D}$ be the proper transform of D , $\hat{\overline{D}}$ the proper transform of $\overline{D}$, and $\hat{F}$ the proper transform of the fundamental system. Then

$$\hat{F} = -\tfrac{1}{2}K - Q_C - Q_{\overline{C}}.$$

The associated map of $|\hat{F}|$, $\hat{\mathbf{\Phi}}$, is a holomorphic map from $\hat{Z}$ onto Q. The following proposition is the key to describe the structure of $\hat{Z}$:

Proposition 3.5. $\mathbf{Q}_C\mathbf{Q}_{\overline{C}}^{-1} \otimes_j \hat{\mathbf{D}}_j := \mathbf{Q}_C\mathbf{Q}_{\overline{C}}^{-1} \otimes \hat{\mathbf{D}}_1 \otimes \cdots \otimes \hat{\mathbf{D}}_\tau$ *is isomorphic to the pullback bundle:* $\hat{\mathbf{\Phi}}^*\mathscr{O}(1, \tau-1)$, *where* $\mathscr{O}(1, \tau-1)$ *is the line bundle on* Q *with bidegree* $(1, \tau-1)$.

Proof. It is enough to show that the line bundle $\mathbf{Q}_C\mathbf{Q}_{\overline{C}}^{-1} \otimes_j \hat{\mathbf{D}}_j$ is trivial on every fiber of the map $\hat{\mathbf{\Phi}}$ and then to compute the bidegree.

We are going to describe the restriction of the map $\hat{\mathbf{\Phi}}$ to the proper transform of any element D in the pencil $|D|$. Let L be the unique real twistor line on D. Then the intersection number LC on the surface D is equal to 1 and hence the curves L and C intersect transversely at one point, say z. By the reality of L, it intersects the conjugate curve $\overline{C}$ at one point $\overline{z}$ on the surface D. Note that $\overline{C}$ cannot be contained in the surface D for otherwise any real twistor line through C would have been contained in D. Moreover, as $\overline{C}$ is the base locus of the pencil $|\overline{D}|$, the intersection number $D\overline{C}$ on the twistor space is equal to $D\overline{D}^2$, which is equal to 1. Therefore, the curve $\overline{C}$ intersects any D transversely at one point. When the curves C and $\overline{C}$ are blown up, the proper transform $\hat{D}$ of the surface D is the blow-up of D at $\overline{z}$. Let the exceptional divisor of blowing-up on $\hat{D}$ be E. Note also that as every divisor $D_{j|D}$ on the surface D is equal to $C + E_j$, it has the curve C as a component. In particular, each $D_{j|D}$ passes through the point z with multiplicity 1.

Therefore, all the divisors $\overline{D}_{j|D}$ pass through the point $\overline{z}$ with multiplicity 1. Then on the surface $\hat{D}$,

$$\hat{D}_{j|\hat{D}} = E_j, \quad \hat{\overline{D}}_{j|\hat{D}} = H - E_j - E,$$

while on D, the fundamental system is $(D + \overline{D})_{|D}$, i.e. $C + H$. Then on $\hat{D}$,

$$\hat{F}_{|\hat{D}} = (C + H) - C - E = H - E.$$

Note that $|H - E|$ is the proper transform of a complete system of lines through the point $\overline{z}$ on $\mathbf{P}^2$. Its associated map is to exhibit $\hat{D}$ as the blow-up of a rational ruled surface with $H - E$ as a generic fiber and E as the ruling. In particular $\hat{\mathbf{\Phi}}$ sends $\hat{D}$ onto a line in Q such that E is mapped onto this line. It is now obvious that any fiber of the map $\hat{\mathbf{\Phi}}$ is a fiber of the restriction of $\hat{\mathbf{\Phi}}$ onto a $\hat{D}$.

We shall prove that the bundle given is trivial on each fiber on $\hat{D}$. On $\hat{D}$, the divisor class of the given line bundle is

$$C - E + \sum_j \hat{D}_{j|\hat{D}} = C - E + \sum_j E_j.$$

An irreducible fiber of the associated map is $H - E$, which is a smooth rational curve. As

$$C(H - E) = 1, \quad E(H - E) = 1, \quad E_j(H - E) = 0;$$

the restriction onto any irreducible fiber is trivial.

Note that components of reducible fibers are E_i, $H - E_i - E$ on a generic $\hat{D}$. If $\hat{D}$ is not generic, there may also be $E_i - E_j$. All of them are smooth rational curves. But

$$CE_i = 1, \quad EE_i = 0;$$

$$(H - E_i - E)C = 0, \quad (H - E_i - E)E = 1, \quad (H - E_i - E)\sum_k E_k = 1;$$

$$(E_i - E_j)C = 0, \quad (E_i - E_j)E = 0, \quad (E_i - E_j)\sum_k E_k = 0;$$

the restriction onto any fiber is trivial.

To compute the bidegree, note that $\hat{\mathbf{\Phi}}$ is a biholomorphism from the pair E and $\overline{E}$ onto the pair of generators of lines on the quadric Q. Moreover,

$$\left(Q_C - Q_{\overline{C}} + \sum_j \hat{D}_j\right)E = \left(C - E + \sum_j \hat{D}_{j|\hat{D}}\right)E$$

$$= \left(C - E + \sum_j E_j\right)E = 1;$$

$$\left(Q_C - Q_{\overline{C}} + \sum_j \hat{D}_j\right)\overline{E} = \left(Q_{\overline{C}} - Q_C + \sum_j \hat{\overline{D}}_j\right)E$$
$$= \left(E - C + \sum_j \hat{\overline{D}}_{j|\hat{D}}\right)E = EE - CE + \sum_j (H - E_j - E)E$$
$$= -1 - 0 + \tau = \tau - 1.$$

Proposition 3.6. *There is a degree-one holomorphic map from* $\hat{Z}\backslash Q_C \cup Q_{\overline{C}}$ *into the variety* $Z_0 := \{(x, y) \in \mathscr{O}(1, \tau - 1) \oplus \mathscr{O}(\tau - 1, 1) : xy = P_1 \cdots P_\tau\}$ *where* P_j *is a section of* $\mathscr{O}(1, 1)$ *over the quadric whose pull-back onto* $\hat{Z}$ *has divisor* $\hat{D}_j + \hat{\overline{D}}_j$.

Proof. Note that all the bundles $\mathbf{Q}_C$, $\mathbf{Q}_{\overline{C}}$, $\hat{\mathbf{D}}_j$, $\hat{\overline{\mathbf{D}}}_j$ have a unique effective divisor on $\hat{Z}$. Therefore, up to a constant, the bundle $\mathbf{Q}_C\mathbf{Q}_{\overline{C}}^{-1} \otimes_j \hat{\mathbf{D}}_j$ and its conjugate have a distinguished meromorphic section, say $\hat{x}$, $\hat{y}$. By Proposition (3.5), $\hat{x}\hat{y}$ is a holomorphic section of $\hat{\mathbf{\Phi}}^*\mathscr{O}(\tau, \tau)$ whose divisor is $\sum_j(\hat{D}_j + \hat{\overline{D}}_j)$. By construction, if x, y are meromorphic sections of $\mathscr{O}(1, \tau - 1)$ and $\mathscr{O}(\tau - 1, 1)$ respectively so that $\hat{\mathbf{\Phi}}^* x = \hat{x}$ and $\hat{\mathbf{\Phi}}^* y = \hat{y}$, then

$$xy = P_1 \cdots P_\tau,$$

where P_j is a section of a holomorphic line bundle on the quadric whose pullback has divisor $\hat{D}_j + \hat{\overline{D}}_j$. To compute the bidegree, we simply note that

$$\hat{D}_j E = \hat{D}_{j|\hat{D}} E = E_j E = 0$$

and

$$\hat{D}_j \overline{E} = \hat{\overline{D}}_j E = \hat{\overline{D}}_{j|\hat{D}} E = (H - E_j - E)E = 1.$$

To finish the proof of the proposition, recall that the restriction of $\mathbf{Q}_C\mathbf{Q}_{\overline{C}}^{-1} \otimes_j \hat{\mathbf{D}}_j$ onto any fiber of $\hat{\mathbf{\Phi}}$ is trivial as the bundle was proved to be a pullback bundle. Therefore, the section $\hat{x}$ restricted onto such fiber is a meromorphic function on a copy of a smooth rational curve with one simple zero and one simple pole. With the pole removed one has a degree 1 holomorphic function. The meromorphic section $\hat{y}$ has the same property. Therefore, $(\hat{x}(z), \hat{y}(z))$ defines the holomorphic map mentioned in the proposition. q.e.d.

The map described in Proposition (3.6) can be holomorphically extended if the target is compactified to $\mathbf{P}(\mathscr{O}(1, \tau - 1) \oplus \mathscr{O}(\tau - 1, 1) \oplus \mathscr{O})$ and we extend the map $(\hat{x}(z), \hat{y}(z))$ to be $[\hat{x}(z), \hat{y}(z), 1]$. It shows that the twistor space is precisely the one constructed by LeBrun [13]. This is the claim of Theorem (3.1). The proof of this claim is now completed.

4. A class of algebraic dimension 1 twistor spaces

Since the description of the algebraic structure of twistor spaces on $\tau\mathbf{P}^2$ with $\tau \leq 3$ is complete. From now on, we shall only consider the case when $\tau \geq 4$. After Theorem (3.1), we turn our attention to the situation when there are effective elementary divisors such that the complete linear system of any such divisor has at most one element. Again, after appropriate choice of basis in the second integral cohomology on the 4-manifold X, we assume that $\dim |D| = 0$ and that for any $j = 1, \cdots, \tau$, the system $|D_j|$ is either empty or is zero-dimensional.

Under the above assumptions, the exact sequence (3.2) implies that $D_{|D}$ is not an effective divisor on the surface D. Then Lemma (1.10) implies that the map ψ_L is not a blow-up of $\mathbf{P}^2$ at any collection of collinear points. In this section, we shall study the algebraic dimension of the twistor space on which the effective elementary divisor D is a blow-up of $\mathbf{P}^2$ at a collection of points such that all but one of them are collinear.

Lemma 4.1. *Suppose that D is an effective divisor of the elementary line bundle $\mathbf{D}$. If the map ψ_L exhibits D as a blow-up of $\mathbf{P}^2$ τ-many times, $\tau \geq 4$, such that all but the last point of blowing-up is collinear, then*:

(1) *the fundamental system on the twistor space has precisely two distinct reducible elements, namely $D + \overline{D}$ and $D_\tau + \overline{D}_\tau$;*

(2) $h^0(Z, \mathscr{K}^{-1/2}) \geq 2$;

(3) *a generic real element of the fundamental system is nonsingular irreducible.*

Proof. After Lemma (1.9), the only possible reducible elements are the sums of effective elementary divisors and their conjugates. By Lemma (1.10), none of the $D_{j|D}$ except $D_{\tau|D}$ is effective.

On the twistor space, due to Hitchin's vanishing theorem [9],

$$h^i(Z, \mathscr{D}_j\mathscr{D}^{-1}) = 0,$$

except possibly when $i = 1$. Then the Riemann-Roch formula shows that $h^1(Z, \mathscr{D}_j\mathscr{D}^{-1}) = 0$. Therefore, the exact sequence

$$0 \to \mathscr{D}_j\mathscr{D}^{-1} \to \mathscr{D}_j \to \mathscr{D}_{j|D} \to 0$$

induces an isomorphism

$$H^0(Z, \mathscr{D}_j) \cong H^0(D, \mathscr{D}_j).$$

Hence, none of the D_j except D_τ is an effective divisor on the twistor space. From the intersection numbers, one can see that $D+\overline{D}$ and $D_\tau+\overline{D}_\tau$ are two distinct elements in the fundamental system. Therefore, (1) is proved. (2) is an obvious consequence of (1).

As the fundamental system has only two reducible elements and there is no fixed component, a generic element of the fundamental system is irreducible. Let S be an irreducible element of the fundamental system. By Bertini's second theorem [1], S can be singular only at the base locus of the system. In particular, it can be singular only at the curves $C := D \cap D_\tau$, $A := D \cap \overline{D}_\tau$ and their conjugate curves $\overline{C}$ and $\overline{A}$. By Lemma (1.10), one can deduce that the curves C and A and their conjugate curves are not real twistor lines. Yet [9, Proposition 4.3(iii)] shows that if S is also real, it can be singular only along some real twistor lines. Therefore, S is nonsingular as claimed in (3). q.e.d.

Theorem 4.2. *Suppose that the twistor space of $\tau\mathbf{P}^2$, $\tau \geq 5$, contains an effective elementary divisor D. If the map ψ_L exhibits D as a blow-up of $\mathbf{P}^2$ τ-times such that all but the last point of blowing-up are collinear, then the algebraic dimension of the twistor space Z is equal to one; i.e. $a(Z) = 1$.*

Proof. With the given configuration of blowing-up on the surface D, Proposition (4.1) shows that $h^0(Z, \mathscr{K}^{-1/2}) \geq 2$. Therefore, $a(Z) \geq 1$.

According to [15, Lemma 2], it suffices to show that $h^0(Z, \mathscr{K}^{-\frac{n}{2}})$ grows as a polynomial in n at most to degree 1.

Let S be a generic real element of the fundamental system. It is a nonsingular irreducible surface in the twistor space as claimed in the last lemma. When we use the map ψ_L of (1.10) to describe the surface D, the intersection of S and D can be considered as an element of the linear system $|2H - \sum_j E_j|$ on D. As all but the τth point are collinear, the intersection is the sum of two curves, namely C and A where C is the proper transform of the line through all but the last point of blowing-up and A is the proper transform of a line through the last point of blowing-up.

Let s be a real section of $\mathscr{K}^{-\frac{n}{2}}$, $n \geq 1$, so that its restriction onto S is not identically zero. Let k and l be the order of vanishing of the restriction of s on S along C and A respectively. By reality, $s_{|S}$ vanishes along $\overline{C}$ and $\overline{A}$ to the same orders. Then

$$J := -nK_S - k(C + \overline{C}) - l(A + \overline{A}) \tag{4.3}$$

is an effective divisor if it is not linearly equivalent to zero.

As

$$-K_S C = -\tfrac{1}{2}KC = -\tfrac{1}{2}K_{|D}C$$
$$= \Big(2H - \sum_j E_j\Big)\Big(H - \sum_j^{\tau-1} E_j\Big) = 3-\tau,$$

and similarly, $-K_S A = 1$. Then by reality,

$$-K_S C = -K_S \overline{C} = 3-\tau, \quad -K_S A = -K_S \overline{A} = 1. \tag{4.4}$$

Since the curves C, $\overline{C}$, A and $\overline{A}$ are all nonsingular rational curves, their self-intersection numbers on the surface S can be computed when we apply the adjunction formula and (4.4). On S, we have

$$C^2 = \overline{C}^2 = 1-\tau, \quad A^2 = \overline{A}^2 = -1. \tag{4.5}$$

Moreover, on the surface S, $-K_S = D_{|S} + \overline{D}_{|S} = C + A + \overline{C} + \overline{A}$. As $c_1^3(Z) = 16(4-\tau)$,

$$2(4-\tau) = (-K_S)^2 = (C + A + \overline{C} + \overline{A})^2. \tag{4.6}$$

Notice that C is a curve on the surface D while D contains a unique real twistor line L such that $CL = 1$; the curve C is disjoint from its conjugate. Similarly, A is disjoint from its conjugate. As we also know, from the configuration on the surface D, that C and A intersect transversely at one point, it follows from (4.5) and (4.6) that $A\overline{C} + \overline{A}C = 2$. By reality, we have $A\overline{C} = \overline{A}C = 1$. With all these intersection numbers available, we can use (4.3) to compute:

$$\begin{aligned} JC &= n(-K_S C) - kC^2 - l(AC + \overline{A}C) \\ &= n(3-\tau) - k(1-\tau) - 2l \\ &= 2(n-l) + (n-k)(1-\tau); \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} JA &= n(-K_S A) - k(CA + \overline{C}A) - lA^2 \\ &= n - 2k + l \\ &= 2(n-k) - (n-l). \end{aligned} \tag{4.8}$$

By the definition of k and l, JC and JA are nonnegative. It follows that

$$2(n-l) \geq (n-k)(\tau - 1), \qquad 2(n-k) \geq (n-l). \tag{4.9}$$

When $\tau \geq 6$, (4.9) is possible only when $n = k = l$ and hence $J = 0$. When $\tau = 5$, (4.9) is possible when

$$JC = J\overline{C} = 0, \qquad JA = J\overline{A} = 0$$

and hence J has no zeros along $C, \overline{C}, A$ and $\overline{A}$. Therefore, the order of zeros of s is constant along $C, \overline{C}, A$, and $\overline{A}$. However, C and A intersect at one point, and thus the orders of zeros of s along C and A are equal. In particular, $k = l$. Then (4.9) is possible only when $k = l = n$.

Therefore, we can conclude that the image of the restriction map

$$H^0(Z, \mathbf{K}^{-\frac{n}{2}}) \to H^0(S, \mathbf{K}_S^{-n})$$

induced by the exact sequence

$$0 \to \mathscr{K}^{-\frac{n-1}{2}} \to \mathscr{K}^{-\frac{n}{2}} \to \mathscr{K}_S^{-n} \to 0$$

is 1-dimensional. Thus,

$$h^0(Z, \mathscr{K}^{-\frac{n}{2}}) \leq h^0(Z, \mathscr{K}^{-\frac{n-1}{2}}) + 1 \leq n + 1.$$

Hence, $a(Z) \leq 1$.

5. Blowing-up of $\mathbf{P}^2$ at co-conic noncollinear points

In this section, we study the blowing-up of at least five points in a position more general than the configurations discussed in Theorem (3.1) and Theorem (4.2). In fact, with the result of Theorem (4.2), the following observation is an example of the semi-continuity principle [8]:

Theorem 5.1. *Suppose that the twistor space of $\tau\mathbf{P}^2$, $\tau \geq 5$, contains an effective elementary divisor D. If the map ψ_L exhibits D as a blow-up of $\mathbf{P}^2$ τ-times such that either all the points of blowing-up are on a non-singular conic or all the points of blowing-up are on two lines such that each line passes through at least two points of blowing-up, then $a(Z) \leq 1$.*

The proof of this theorem relies on two technical lemmata:

Lemma 5.2. *Let D be a blow-up of $\mathbf{P}^2$ at points on an irreducible conic or a sum of two lines as described in the hypothesis of Theorem* (5.1). *Then* $h^0(D, \mathscr{K}_{|D}^{-\frac{n}{2}}) = 1$.

Lemma 5.3. *With the hypothesis of Theorem* (5.1), *the following exact sequence on the twistor space*

$$(5.4) \qquad 0 \to \mathscr{K}^{-(\frac{n-1}{2})} \to \overline{\mathscr{D}}^{n-1}\mathscr{D}^n \xrightarrow{\rho} \overline{\mathscr{D}}^{n-1}\mathscr{D}^n_{|D} \to 0$$

induces a natural isomorphism

$$H^0(Z, \mathscr{K}^{-(\frac{n-1}{2})}) \cong H^0(Z, \overline{\mathscr{D}}^{n-1}\mathscr{D}^n).$$

Proof of Theorem 5.1. Considering the exact sequence

$$0 \to \mathscr{D}^{n-1}\overline{\mathscr{D}}^n \to \mathscr{K}^{-\frac{n}{2}} \xrightarrow{r} \mathscr{K}_{|D}^{-\frac{n}{2}} \to 0, \tag{5.5}$$

we have

$$h^0(Z, \mathscr{K}^{-\frac{n}{2}}) = h^0(Z, \mathscr{D}^{n-1}\overline{\mathscr{D}}^n) + \dim \text{ image of } r.$$

By reality,

$$h^0(Z, \mathscr{K}^{-\frac{n}{2}}) = h^0(Z, \mathscr{D}^n\overline{\mathscr{D}}^{n-1}) + \dim \text{ image of } r.$$

Using Lemma (5.2), we obtain

$$h^0(Z, \mathscr{K}^{-\frac{n}{2}}) \le h^0(Z, \mathscr{D}^n\overline{\mathscr{D}}^{n-1}) + 1.$$

From Lemma (5.3) it follows that

$$\begin{aligned} h^0(Z, \mathscr{K}^{-n/2}) &\le h^0(Z, \mathscr{K}^{-(n-1)/2}) + 1 \\ &\le \underbrace{1 + \cdots + 1}_{n \text{ times}} + h^0(Z, \mathscr{O}) = n + 1. \end{aligned}$$

As a consequence of [15, Lemma 2], $a(Z) \le 1$.

Proof of Lemma 5.2. In this proof, we have to consider two different possibilities, namely when the conic is irreducible and when the conic is the sum of two lines such that each line passes through at least two of the blown-up points. We shall treat the case when the conic is irreducible first.

Due to Lemma (1.10), $-\frac{n}{2}K_{|D} = n(2H - \sum_{i=1}^{\tau} E_i)$. Therefore, when D is the blow-up at τ points, $\tau \ge 5$, on an irreducible conic, the divisor class $2H - \sum_{i=1}^{\tau} E_i$ is represented by an irreducible rational curve with negative self-intersection. Therefore $h^0(D, \mathscr{K}_{|D}^{-\frac{n}{2}}) = 1$.

When the points of blowing up are on two lines so that each line contains at least two points, let C and A be the proper transform of the two lines so that the number of points on C is not smaller than the number of points on A. In particular,

$$C = H - \sum_{i=1}^{j} E_i, \qquad A = H - \sum_{i=j+1}^{\tau} E_i,$$

where $j \ge 3$, $2j \ge \tau$. Let k and l be the order of vanishing of a section of $\mathscr{K}_{|D}^{-\frac{n}{2}}$ along C and A respectively. Then

$$J = n(C + A) - kC - lA = (n - k)C + (n - l)A$$

is an effective divisor not containing C and A if it is not identically zero. As

$$JC = (n-k)(1-j) + (n-l), \quad JA = (n-k) + (n-l)(1-(\tau-j)),$$
$$(n-l) \geq (j-1)(n-k), \quad (n-k) \geq ((\tau-j)-1)(n-l).$$

Then $(n-k) \geq ((\tau-j)-1)(j-1)(n-k)$. As $j \geq 3$, when $(\tau-j) \geq 2$, it is possible only when $n=k=l$ and hence J is identically zero. Therefore $h^0(D, \mathscr{K}_{|D}^{-\frac{n}{2}}) = 1$.

Proof of Lemma 5.3. To prove this claim, let t be a section of $\overline{\mathbf{D}}^{n-1}\mathbf{D}^n$; then the conjugate section $\bar{t}$ is a section of $\mathbf{D}^{n-1}\overline{\mathbf{D}}^n$. Therefore, $t\bar{t}$ is a section of $\overline{\mathbf{D}}^{n-1}\mathbf{D}^n\mathbf{D}^{n-1}\overline{\mathbf{D}}^n$, i.e. a section of $\mathbf{K}^{-(\frac{2n-1}{2})}$. According to Lemma (5.2), if the restriction of $t\bar{t}$ on D, $r(t\bar{t})$, is not identically zero, the divisor is $(2n-1)(C+A)$. In this case, the divisor of $\rho(t)$ on D is $kC + lA$ for some positive integers k and l. But there are no such integers so that

$$k\left(H - \sum_{i=1}^{\tau-1} E_i\right) + l(H - E_\tau) = n\left(H - \sum_{i=1}^{\tau} E_i\right) + (n-1)H.$$

Therefore, $r(t\bar{t})$ is identically zero.

Either t or $\bar{t}$ vanishes identically on D. If $\bar{t}$ vanishes identically on D, let its order of zeros along D be m. When d is a section of the bundle of $\mathbf{D}$ so that its divisor is D, then there is a section $\bar{u}$ of $\mathbf{D}^{n-1-m}\overline{\mathbf{D}}^n$ such that $\bar{t} = \bar{u}d^m$. Then $t = u\bar{d}^m$ and u is a section of $\overline{\mathbf{D}}^{n-1-m}\mathbf{D}^n$. The previous argument on $t\bar{t}$ is now applied to $u\bar{u}$ to conclude that $u\bar{u}$ vanishes identically on D. By the definition of m, $\bar{u}$ cannot vanish identically on D. Therefore, u has to vanish identically on D and hence t always vanishes identically on D. It means that the restriction map ρ is the zero map. q.e.d.

Combining Theorem (4.2) and Theorem (5.1), we can arrive at

Theorem 5.6. *Suppose that the twistor space of* $\tau\mathbf{P}^2$, $\tau \geq 5$, *contains an effective elementary divisor* D. *If the map* ψ_L *exhibits* D *as a blow-up of* $\mathbf{P}^2$ τ*-times such that all the points of blowing-up are co-conic noncollinear, then* $a(Z) \leq 1$.

6. Blowing-up of $\mathbf{P}^2$ at points in general positions

To finish our discussion of the algebraic dimension of twistor space of $\tau\mathbf{P}^2$, $\tau \geq 5$, admitting an effective elementary divisor, we shall examine the situation when the effective elementary divisor is a blow-up of $\mathbf{P}^2$ at generic points. When $\tau = 5$, the generic configuration is already studied in Theorem (5.1) because through five generic points, there is a nonsingular conic. Therefore except in Lemma (6.1), we shall always assume that $\tau \geq 6$ throughout this section.

Lemma 6.1. *Suppose that the map ψ_L on D is a blow-up of $\mathbf{P}^2$ at τ-many points, $\tau \geq 4$, such that no four of them are collinear. Then*

$$H^0(Z, \mathscr{K}^{-\frac{n}{2}}) \cong H^0(Z, \mathscr{K}^{-\frac{n}{2}} D)$$

for all $n \geq 0$.

Proof. By the exact sequence

$$0 \to \mathscr{K}^{-\frac{n}{2}} \to \mathscr{K}^{-\frac{n}{2}}\mathscr{D} \to (\mathscr{K}^{-\frac{n}{2}}\mathscr{D})_{|D} \to 0,$$

it suffices to show that $h^0(D, (\mathscr{K}^{-\frac{n}{2}}\mathscr{D})_{|D}) = 0$. Recall Lemma (1.10) that

$$(-\tfrac{n}{2}K + D)_{|D} = n(2H - E_1 - \cdots - E_\tau) + (H - E_1 - \cdots - E_\tau).$$

Its intersection with $H - E_1 - E_2$ and $H - E_3 - E_4$ is negative. Therefore, if it were effective, then for all positive integer n,

$$(n-1)(2H - E_1 - E_2 - E_3 - E_4) - n(E_5 + \cdots + E_\tau) + (H - E_1 - \cdots - E_\tau)$$

would have been effective. Inductively, we can conclude that $-n(E_5 + \cdots + E_\tau) + (H - E_1 - \cdots - E_\tau)$ is effective. This is a contradiction to the hypothesis when $n = 0$. This is absurd when n is positive.

Lemma 6.2. *When the map ψ_L on the surface D is a blow-up of $\mathbf{P}^2$ at least at six points such that no three of them are collinear and no six of them are co-conic, then $h^0(D, \mathscr{K}_{|D}^{-\frac{n}{2}}) = 0$.*

Proof. Since $-\frac{1}{2}K_{|D} = 2H - \sum_j E_j$, the statement of this lemma is a classical result in algebraic geometry, which can be proved, for instance, by the method of the proof of the last lemma. q.e.d.

After these two lemmata, we are ready to prove the following theorem.

Theorem 6.3. *When the map ψ_L on the surface D is a blow-up of $\mathbf{P}^2$ at least at six points such that no three of them are collinear and no six of them are co-conic, then the algebraic dimenion of the twistor space is equal to zero.*

Proof. The induced long exact sequence of

$$(6.4) \qquad 0 \to \mathscr{K}^{-\frac{n-1}{2}}\overline{\mathscr{D}} \to \mathscr{K}^{-\frac{n}{2}} \to \mathscr{K}_{|D}^{-\frac{n}{2}} \to 0$$

implies that

$$h^0(Z, \mathscr{K}^{-\frac{n}{2}}) \leq h^0(Z, \mathscr{K}^{-\frac{n-1}{2}}\overline{\mathscr{D}}) + h^0(D, \mathscr{K}_{|D}^{-\frac{n}{2}}).$$

By reality, this inequality is equivalent to

$$h^0(Z, \mathscr{K}^{-\frac{n}{2}}) \le h^0(Z, \mathscr{K}^{-\frac{n-1}{2}}\mathscr{D}) + h^0(D, \mathscr{K}_{|D}^{-\frac{n}{2}}).$$

Then Lemma (6.2) implies that

$$h^0(Z, \mathscr{K}^{-\frac{n}{2}}) \le h^0(Z, \mathscr{K}^{-\frac{n-1}{2}}\mathscr{D}).$$

According to Lemma (6.1), this inequality is equivalent to

$$h^0(Z, \mathscr{K}^{-\frac{n}{2}}) \le h^0(Z, \mathscr{K}^{-\frac{n-1}{2}}).$$

Therefore,

$$h^0(Z, \mathscr{K}^{-\frac{n}{2}}) \le 1.$$

As the fundamental line bundle does have nontrivial holomorphic section, namely, the one whose zero divisor is the sum of the effective elementary divisor D and its conjugate with multiplicity n,

$$h^0(Z, \mathscr{K}^{-\frac{n}{2}}) = 1.$$

By [15, Lemma 2], $a(Z) = 0$.

7. Algebraic dimension of twistor spaces of $4\mathbf{P}^2$

As we remarked in the introduction, the algebraic structure of twistor spaces associated to $4\mathbf{P}^2$ requires a seperate treatment. Throughout this section, we assume that $\tau = 4$. As we always do, we also assume that the elementary line bundle $\mathbf{D}$ has an effective divisor D such that $\dim|D| \ge \dim|D_j|$. With respect to the blow-up map ψ_L of Lemma (1.10), the blow-up points can be collinear as we had discussed in §3. We shall study the remaining two configurations of blowing-up, namely when three of the four points are collinear and when the four points are in general position.

Note that when the self-dual conformal class on $4\mathbf{P}^2$ contains a metric of positive scalar curvature, the Hitchin's vanishing theorems [9] and the Riemann-Roch formula imply that

$$h^0(Z, \mathscr{K}^{-\frac{1}{2}}) - h^1(Z, \mathscr{K}^{-\frac{1}{2}}) = 2. \tag{7.1}$$

In particular, $a(Z) \ge 1$.

Theorem 7.2. *Suppose that the elementary line bundle* $\mathbf{D}$ *on a twistor space of* $4\mathbf{P}^2$ *has effective divisor* D. *If the map* ψ_L *on* D *is a blow-up of* $\mathbf{P}^2$ *at four points in general position, then* $a(Z) \le 2$.

Proof. The proof of this theorem is similar to the proof of Theorem (6.3). In fact, we can apply exact sequence (6.4) as well as Lemma (6.1).

The only difference is that in the proof of (6.3), one has Lemma (6.2), which states that $h^0(D, \mathscr{K}_{|D}^{-\frac{n}{2}}) = 0$, while the present configuration of blowing-up yields $h^0(D, \mathscr{K}_{|D}^{-\frac{n}{2}}) = n+1$.

The conclusion is that $h^0(Z, \mathscr{K}^{-\frac{n}{2}})$ grows as a polynomial in n at most quadratically. Then the proof of the theorem is finished by [15, Lemma 2]. q.e.d.

To deal with the case when three of the four points of blowing-up are collinear, we recall Lemma (4.1) that a generic real element of the fundamental system is nonsingular irreducible. We shall examine the algebraic structure of such a fundamental divisor and then study the canonical system of the twistor space.

By Lemma (1.10), the restriction of any effective fundamental divisor onto the surface D is an element of the system of conics through the four points of blowing-up. As we have seen in the proof of Theorem (4.3), when three of the four points are collinear, such a divisor on the surface D is the sum of two irreducible nonsingular curves C and A where C is the proper transform of the line through the three collinear points and A is the proper transform of the line through the remaining point of blowing-up. On the conjugate surface $\overline{D}$, one finds the conjugate curves $\overline{C}$ and $\overline{A}$. The calculation from (4.4) to (4.6) shows that

$$(7.3) \qquad -\tfrac{1}{2}KC = -\tfrac{1}{2}K\overline{C} = -1, \qquad -\tfrac{1}{2}KA = -\tfrac{1}{2}K\overline{A} = +1;$$

and the conjugate pair C and $\overline{C}$ are mutually disjoint. Also, the conjugate pair A and $\overline{A}$ are mutually disjoint. Yet both A and $\overline{A}$ intersect both C and $\overline{C}$ transversely at one point. When S is a generic real fundamental divisor, S is nonsingular irreducible by Lemma (4.1). With the above notations, we shall prove the following lemma:

Lemma 7.4. *When S is a generic real fundamental divisor, then S is a blow-up of $\mathbf{P}^2$ nine times. Let H be the hyperplane class on $\mathbf{P}^2$ and E_i the ith exceptional divisor of blowing-up. Then*

$$A = E_8, \qquad \overline{A} = E_9,$$

$$(7.5) \qquad C = 3H - \sum_{i=1}^{6} E_i - 2E_7 - E_8 - E_9, \qquad \overline{C} = E_7 - E_8 - E_9.$$

Moreover, the first seven points of blowing-up are not infinitely near each other.

Remark 7.6. The result of Lemma (7.4) means that C is the proper transform of an irreducible cubic with a node. The surface S is obtained by blowing-up $\mathbf{P}^2$ at seven distinct points on the cubic including the node and then, on each of the two tangent lines to the cubic at the node, blowing-up one point infinitely near the node.

Proof of Lemma 7.4. By Lemma (1.3), the surface S is the blowing-up of a rational ruled surface eight times. Therefore, if S can be obtained by blowing-up $\mathbf{P}^2$, it is a blowing-up of $\mathbf{P}^2$ nine times.

Note that if E is an irreducible curve in the twistor space such that $-\frac{1}{2}KE < 0$, then E is in the base locus of the fundamental system. In particular, E or its conjugate is contained in the intersection of D and a generic fundamental divisor S. With the description (1.10), we can conclude that E is either the curve C or $\overline{C}$.

If E is an irreducible curve in the twistor space such that $-\frac{1}{2}KE = 0$, then either E is in the base locus of the fundamental system or a generic fundamental divisor is disjoint from E. With the given configuration of blowing-up on the surface D, the fundamental system has no such curve as base locus because E is neither C nor linearly equivalent to A on the surface D. Since by the adjunction formula, $-K_S = -\frac{1}{2}K_{|S}$, we conclude from this and the last paragraph that on a generic real fundamental divisor S,

$$-K_S F \geq 1 \tag{7.7}$$

for any irreducible curve F on the surface S except when the curve is either C or $\overline{C}$.

By (7.3), A and $\overline{A}$ are (-1)-curves. As they are mutually disjoint, both of them can be blown down to a point. After they are blown down, C and $\overline{C}$ are a pair of (-1)-curves intersecting transversely at two distinct points. Since C and $\overline{C}$ are the only irreducible curves on S such that (7.7) fails to hold, after A and $\overline{A}$ are blown down, one obtains a del Pezzo surface. Since $(-K_S)^2 = 0$, this Del Pezzo is a blow-up of $\mathbf{P}^2$ at seven distinct points. Up to a choice of notation, we can assume that $\overline{C}$ is blown down to a point. If we set

$$A = E_8, \qquad \overline{A} = E_9,$$

then

$$\overline{C} = E_7 - E_8 - E_9.$$

Since the anticanonical divisor on S is linearly equivalent to $(D + \overline{D})_{|S}$, which is $C+\overline{C}+A+\overline{A}$,

$$\begin{aligned} C &= -K_S - \overline{C} - A - \overline{A} \\ &= \left(3H - \sum_{i=1}^{9} E_i\right) - (E_7 - E_8 - E_9) - E_8 - E_9 \\ &= 3H - \sum_{i=1}^{6} E_i - 2E_7 - E_8 - E_9. \quad \text{q.e.d.} \end{aligned}$$

Due to (7.3), C and $\overline{C}$ are in the base locus of the anticanonical system of the surface S. As

$$-K_S - C - \overline{C} = E_8 + E_9,$$

the only effective anticanonical divisor on S is $C+\overline{C}+A+\overline{A}$. Therefore $h^0(S, \mathscr{K}_S^{-1}) = 1$. As the exact sequence

$$0 \to \mathscr{O} \to \mathscr{K}^{-\frac{1}{2}} \to \mathscr{K}_S^{-1} \to 0$$

induces an exact sequence of 0th cohomology, we can conclude that

$$h^0(Z, \mathscr{K}^{-\frac{1}{2}}) = 2.$$

With this equality, the Riemann-Roch theorem and Hitchin's vanishing theorem together imply that $h^1(Z, \mathscr{K}^{-\frac{1}{2}}) = 0$ when $\tau = 4$. Therefore, we have the following exact sequence:

$$(7.8) \qquad 0 \to H^0(Z, \mathscr{K}^{-\frac{1}{2}}) \to H^0(Z, \mathscr{K}^{-1}) \to H^0(S, \mathscr{K}_S^{-2}) \to 0.$$

Due to (7.3), C and $\overline{C}$ are in the base locus of $|-2K_S|$. On S, define $J := -2K_S - (C+\overline{C})$; then from (7.5), we have

$$(7.9) \qquad J = 3H - \sum_{i=1}^{7} E_i .$$

After Lemma (7.4), the complete linear system of J is the system of cubics through the seven distinct points on the blow-down of the curve C. As C is blown down to be an irreducible cubic and these seven points are not co-conic, it follows that the dimension of the complete system $|J|$ is equal to 2, and hence $\dim|-2K_S| = \dim|J| = 2$. With the exact sequence (7.8), we have $h^0(Z, \mathscr{K}^{-1}) = 5$. Moreover, as $|J|$ has no base points, we have the following lemma:

Lemma 7.10. *The anticanonical system is four-dimensional and its base locus are the curves C and $\overline{C}$ with multiplicity one.*

Theorem 7.11. *Suppose that* Z *is a twistor space associated to a self-dual metric of positive scalar curvature on* $4\mathbf{P}^2$. *If* Z *contains an effective elementary divisor such that the map* ψ_L *exhibits* D *as a blow-up of* $\mathbf{P}^2$ *at four points such that three of them are collinear, then* $a(Z) = 3$.

Proof. Let Ψ be the associated map of the anticanonical bundle on the twistor space. Its range, due to Lemma (7.10), is in $\mathbf{P}^4$. As the anticanonical system has C and its conjugate as its base locus, Ψ is a meromorphic map with poles of order one along these two curves. Let $\mathbf{Q}$ and $\overline{\mathbf{Q}}$ be the exceptional divisors of the blow-up of C and $\overline{C}$ on the twistor space and $\hat{\Psi}$ be the associated map of $(\mathbf{KQ\overline{Q}})^{-1}$; then $\hat{\Psi}$ is a holomorphic map from the proper transform of Z, $\hat{Z}$, into $\mathbf{P}^4$. We shall prove the theorem by showing that the image of $\hat{Z}$ via the map $\hat{\Psi}$ is a three-dimensional variety in $\mathbf{P}^4$.

When $\hat{\Psi}$ is restricted onto the proper transform of a fundamental divisor S described in Lemma (7.4), it is precisely the associated map of the complete linear system $|J|$ on S defined in (7.9). Therefore, $\hat{\Psi}_{|\hat{S}}$ is the composition of blowing down E_8 and E_9 and the associated map, say Ψ, of the anticanonical system of a blow-up of $\mathbf{P}^2$ at seven generic points. The map Ψ exhibits this rational surface as a double covering of $\mathbf{P}^2$ branched along a quartic as we have seen in §3. In particular, $\hat{\Psi}(\hat{S})$ is a nonsingular variety biholomorphic to a copy of $\mathbf{P}^2$ in $\mathbf{P}^4$. This shows that the dimension of $\hat{\Psi}(\hat{Z})$ is at least two.

We claim that this copy of $\mathbf{P}^2$ is a linear subspace of $\mathbf{P}^4$. In fact, S is a fundamental divisor. Hence $\hat{S}$ is an element in $|-\frac{1}{2}K - Q - \overline{Q}|$. Then $2\hat{S} + Q + \overline{Q}$ is an element of the anticanonical system of $\hat{Z}$. In particular, $\hat{\Psi}(\hat{S})$ is contained in a hyperplane of $\mathbf{P}^4$. As a copy of $\mathbf{P}^2$ in a hyperplane of $\mathbf{P}^4$, $\hat{\Psi}(\hat{S})$ must be a linear subspace.

On the other hand, the proper transform of a generic element of the anticanonical system on the twistor space is mapped into a subvariety of a hyperplane section of $\mathbf{P}^4$. If $\dim \hat{\Psi}(\hat{Z})$ were equal to two, then this variety would have been contained in the copy of $\mathbf{P}^2$ that we found in the last paragraph. This is impossible because it would imply that the dimension of the anticanonical system was equal to three. Therefore, we can conclude that $\dim \hat{\Psi}(\hat{Z}) = 3$ as claimed. Therefore, $a(Z) = 3$. q.e.d.

Appendix

In this section, we carry on the proof in §2 to show that there is no conjugate pair of singular points on B other than the six pairs on the edges of the tetrahedron.

Assuming on the contrary that such a pair of points exists, the line ℓ passing through them will not intersect B anywhere but p and $\bar{p}$. In particular, a generic hyperplane P containing ℓ will not contain the thirteen points of singularity and the line $P \cap P_j$ intersects L_j transversely at two points. It follows that a generic real S in $|-\frac{1}{2}K|_\ell$ intersects all elementary divisors along irreducible curves. With (1.8), we work out the intersection matrix of these curves on S with respect to the following order: $\{D_0, D_1, D_2, D_3, \overline{D}_0, \overline{D}_1, \overline{D}_2, \overline{D}_3\}$. It is

$$\text{(A.1)} \qquad \begin{pmatrix} -1 & 0 & 0 & 0 & 2 & 1 & 1 & 1 \\ 0 & -1 & 1 & 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 1 & 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 & 0 & 2 \\ 2 & 1 & 1 & 1 & -1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & -1 & 1 & 1 \\ 1 & 0 & 2 & 0 & 0 & 1 & -1 & 1 \\ 1 & 0 & 0 & 2 & 0 & 1 & 1 & -1 \end{pmatrix}.$$

Moreover,

$$K_S D_j = -1, \qquad \text{for all } j.$$

Each D_j or $\overline{D}_j$ is a (-1)-curve.

On the other hand, S contains a real twistor line L for topological reasons [18]. By Lemma (1.3), the associated map of L exhibits S as the blow-up of a Hirzebruch surface, Σ_k, six times. We may assume that the blow-up points are away from the infinity section E_∞. As the fundamental system on the twistor space is base point free, there are no $(-k)$-curves with $k \geq 3$. Therefore, $k = 1$ or $k = 2$.

To express the fundamental divisor in the usual structure on a rational ruled surface, let E_0 be the divisor class of the zero section, F the class of a fiber, E_j the exceptional divisor of the j th blow-up.

Assume that $k = 2$. Then E_∞ is a (-2)-curve. The associated map is going to contract it to a singular point p. In particular,

$$D_j E_\infty = 0, \qquad \overline{D}_j E_\infty = 0.$$

As

$$D_j F = D_j L = 0, \qquad \overline{D}_j F = \overline{D}_j L = 0.$$

From table (A.1), we deduce that

$$\text{(A.2)}\qquad \begin{aligned} D_0 &= E_0 - E_1 - E_2 - E_3, & \overline{D}_0 &= E_0 - E_4 - E_5 - E_6, \\ D_1 &= E_0 - E_2 - E_3 - E_4, & \overline{D}_1 &= E_0 - E_1 - E_5 - E_6, \\ D_2 &= E_0 - E_1 - E_3 - E_5, & \overline{D}_2 &= E_0 - E_2 - E_4 - E_6, \\ D_3 &= E_0 - E_1 - E_2 - E_6, & \overline{D}_3 &= E_0 - E_3 - E_4 - E_5. \end{aligned}$$

To see the real structure on the second cohomology group on S with respect to the exceptional divisors of blowing-up, let us work on E_1. As $E_1 L = 0$, $\overline{E}_1 L = 0$, then

$$\overline{E}_1 = nF - \sum n_i E_i.$$

As E_1 is irreducible, so is $\overline{E}_1$. Since $E_1^2 = \overline{E}_1^2 = -1$ and $D_j \overline{E}_1 = \overline{D}_j E_1$, from the given expression of the D_j and $\overline{D}_j$ in (A.2), we found that

$$\begin{cases} n_1 + n_2 + n_3 = n, \\ n_2 + n_3 + n_4 = n - 1, \\ n_2 = n_5, \\ n_3 = n_6, \\ \sum n_i^2 = 1. \end{cases}$$

As $K_S E_1 = -1$, E_1 is not real. The only possible solution to the above system of equations other than $(n_1, \cdots, n_6) = (1, 0, 0, 0, 0, 0)$ is $(0, 0, 0, 1, 0, 0)$, i.e. $\overline{E}_1 = E_4$. Similarly, one can deduce that $\overline{E}_2 = E_5$ and $\overline{E}_3 = E_6$.

To study E_∞, we first observe that E_∞ is not real because $E_\infty L = 1$. Then there is $\overline{E}_\infty$ with $\overline{E}_\infty L = 1$. Therefore,

$$\overline{E}_\infty = E_0 + nF - \sum n_i E_i.$$

As $E_\infty E_i = 0$, $\overline{E}_\infty E_i = 0$ for $i = 0, 1, \cdots, 6$. Then

$$\overline{E}_\infty = E_0 + nF.$$

Yet

$$-2 = E_\infty^2 = \overline{E}_\infty^2 = (E_0 + nF)^2 = 2 + 2n.$$

Then $n = -2$. This is impossible because $\overline{E}_\infty$ is supposed to be an irreducible nonsingular curve.

The remaining possibility is when $k = 1$. As $D_0 L = 1$, in this case, we may choose D_0 to be the infinity section. Then we deduce from (A.1) that

$$
\begin{aligned}
&D_0 = E_0 - F, && \overline{D}_0 = E_0 + 2F - \sum E_i, \\
&D_1 = E_0 - E_5 - E_6, && \overline{D}_1 = E_0 + F - E_1 - E_2 - E_3 - E_4, \\
&D_2 = E_0 - E_3 - E_4, && \overline{D}_2 = E_0 + F - E_1 - E_2 - E_5 - E_6, \\
\text{(A.3)}\quad &D_3 = E_0 - E_1 - E_2, && \overline{D}_3 = E_0 + F - E_3 - E_4 - E_5 - E_6.
\end{aligned}
$$

Given this data, the method used in the case of $k = 2$ can be applied to show that

$$
\begin{aligned}
\overline{E}_1 &= F - E_2, & \overline{E}_2 &= F - E_1, \\
\overline{E}_3 &= F - E_4, & \overline{E}_4 &= F - E_3, \\
\overline{E}_5 &= F - E_6, & \overline{E}_6 &= F - E_5.
\end{aligned}
$$

Now assume that C is a nonreal (-2)-curve contracting to the point p. Let it be

$$C = nE_0 + mF - \sum n_j E_j.$$

Then $n = CL$. If $n \geq 1$, then $\Phi(L)$ passes through p and $\overline{p}$. This would imply that the line ℓ is the image of real twistor line. In particular, ℓ passes through B at a point other than p and $\overline{p}$, namely, u. This is impossible. Therefore $n = 0$.

As C is a (-2)-curve disjoint from all the D_j and $\overline{D}_j$, with the given expression in (A.3), we found that

$$
\begin{cases}
m = 0, \\
n_1 + n_2 = 0, \\
n_3 + n_4 = 0, \\
n_5 + n_6 = 0, \\
\sum n_i^2 = 2.
\end{cases}
$$

The only three possibilities are

$$E_1 - E_2, \quad E_3 - E_4, \quad E_5 - E_6.$$

Say $C = E_1 - E_2$; then $\overline{C} = E_3 - E_4$ (or $E_5 - E_6$). Thus

$$\overline{C} \cdot \overline{E}_1 = \overline{C}(F - E_2) = (E_3 - E_4)(F - E_2) = 0,$$

while

$$\overline{C} \cdot \overline{E}_1 = CE_1 = (E_1 - E_2)E_1 = -1.$$

This contradiction concludes that such system $|-\frac{1}{2}K|_\ell$ cannot exist.

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